A MANUAL OF PRACTICAL MATHEMATICS

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BY

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PREFACE.

One of the chief objects of this volume is to bring within the reach of students of ordinary abilities, and to enable them to make practical use of, some portions of what are generally, though with little reason, called "higher" Mathematics. Many mathematical rules, such as those studied under Mensuration to obtain the volume and surface of a sphere, may be obtained by so-called "elementary" methods, but these are frequently only roundabout and troublesome tricks, and are after all merely expedients to evade the simple notation of the Calculus and usually end by assuming the idea of a limit, a conception which my experience shows is quite as difficult for the student to grasp as the underlying principles of the Calculus. Or, take the problem of determining the moment of inertia of a rod: when once the student becomes familiar with the easy language of the Calculus, all the scaffolding, which has to be so carefully and tediously built up to obtain a result if Algebra alone is employed, may be at once discarded.

For these and similar reasons, and to keep the size of the book within reasonable limits, the rudiments of Mathematics—Arithmetic and simple Algebra—are taken for granted, though summaries of the more important elementary results are given at the beginning of each section. A student not already familiar with the proofs leading to these results and at home with illustrative examples on them should refer to my earlier books or some similar source. The summaries referred to are in every case followed by concrete numerical examples fully worked out and a set of exercises to enable the student to

become possessed of the full meaning of each of the terms in the algebraic expressions representing the rules.

The order of treatment merely represents what I have found to be most advantageous in my own classes. Other teachers may find it better to vary the sequence to meet the particular requirements of their own students. Readers who are studying without the help of a teacher are recommended to omit the more difficult sections at the first reading. I should like to direct particular attention to several portions of the book, for, so far as I am aware, the method of treatment therein is now published for the first time. Among these sections may be mentioned :

(*a*) The graphical method of solving a quadratic equation.

(*b*) The identification of the nature of a plotted curve by the use of a strip of celluloid on which a series of standard curves is already drawn; and the method of finding the value of n in the family of curves denoted by $y=x^n$, etc.

(*c*) The method of solution of equations of the form $T=a+by^n$.

(*d*) The graphical methods of dealing with problems in Simple Harmonic Motion expressed by $y=a\cos(\omega t+e)$, or $y=a\sin(bx+c)$.

(*e*) The problems involving addition and subtraction of simple solids.

(*f*) The theory of the Amsler planimeter, of vector notation, and of Fourier's theorem. In this connection I am glad to express my grateful indebtedness to Mr. Joseph Harrison, of the Royal College of Science, for portions of the proofs.

(*g*) The graphical method of obtaining the slope of a curve by means of a set-square and pencil.

(*h*) The geometrical proof that $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$.

(*j*) The use of arithmetical and geometrical progressions to illustrate the Integral Calculus.

Great importance has been attached throughout the book to

fully-worked concrete examples, and of these a very large number is to be found; it is hoped that the student will be able, by means of these examples, to follow intelligently every step of his work. The examples and exercises are either original, having been made to illustrate the text, or they have been carefully selected from examination papers, mainly those of the Board of Education. Answers are given to all exercises, and these have been checked with great care, but in so large a number of solutions it is perhaps too much to hope that mistakes have been entirely avoided. I should be grateful to any teacher who would call my attention to any correction which seems necessary.

In order not to overburden the book, I have been compelled to be very brief in some parts, especially in my treatment of the Calculus and of Differential Equations. Students who wish for more detailed information should consult Prof. Perry's *Calculus for Engineers*, where they will find complete guidance in the further study of the subject. Those who know the writings and lectures of Prof Perry, F.R.S., will appreciate how much I owe to his inspiration, and I am glad again to record this debt. The treatment adopted in this book is, however, based always upon my own long experience as a teacher of Mathematics.

In the preparation of my MSS and in the passage of the book through the press I have received much assistance from many friends, whose help I am pleased thus to acknowledge. Mr L. Bairstow has looked through the MSS. and made many valuable suggestions; Mr H J. Woodall has read all the proofs and usefully altered and corrected my work in many places; and Prof. R A. Gregory and Mr. A T. Simmons, B.Sc., have again given me the benefit of their kindly and experienced criticism at every stage in the preparation of the book.

FRANK CASTLE.

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CHAPTER I.

SIMPLIFICATIONS AND PARTIAL FRACTIONS.

Elementary results and formulae.—The following formulae are probably already familiar to the reader. If not, they should, after verification by actual multiplication, be committed to memory ·

$$(a+b)^2 = a^2 + 2ab + b^2\ ;\quad (a-b)^2 = a^2 - 2ab + b^2.$$

The two formulae may be combined, thus.

$$\mathbf{(a \pm b)^2 = a^2 \pm 2ab + b^2.}$$

These formulae should be equally familiar when other letters are used, such as x, y, etc

Ex. 1. $$(3ax - 2ay)^2 = (3ax)^2 - 2 \times (3ax) \times (2ay) + (2ay)^2$$
$$= 9a^2x^2 - 12a^2xy + 4a^2y^2.$$

Similarly, by multiplication,

$$\mathbf{(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.}$$
$$\mathbf{(a+b)(a-b) = a^2 - b^2.}$$

The last example may be expressed in words by saying: **The product of the sum and difference of two quantities is equal to the difference of their squares.**

Ex. 2 $$127^2 - 123^2 = (127 + 123)(127 - 123)$$
$$= 250 \times 4 = 1000.$$

Ex. 3. $$9c^2 - 16(a-b)^2 = (3c)^2 - \{4(a-b)\}^2$$
$$= (3c + 4a - 4b)(3c - 4a + 4b).$$

Square of a polynomial.—The square of an expression consisting of three or more terms can be obtained by arranging the terms as in Multiplication, and obtaining the product; but the work is much reduced by noticing the arrangement of the terms in

$$(a+b)^2=a^2+b^2+2ab,$$

and applying the result to any expression containing three or more terms, it is then easy to write down the square required.

Thus, $\quad (a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$

On the right-hand side the sum of the squares of the three separate terms are followed by twice the products of the first and second, the first and third, and finally of the second and third terms respectively. Similarly,

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$$

Other expressions involving squares and cubes should be written down in a similar manner and verified.

Fractional expressions.—In the simplification of fractional expressions, a factor of the denominator of one fraction may be equal to a factor of another denominator with its sign changed. In such cases, it is advisable to change the sign of one of the fractions by multiplying its numerator and denominator by -1. If any fraction requires two such changes the original sign will remain unaltered.

Ex. 1. Simplify

$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}.$$

Here one change must be made in the second fraction and two changes in the third. Hence, the sign of the second fraction will be altered and the third will remain unchanged By thus altering the second and third fractions, the given expression becomes

$$\frac{a}{(a-b)(a-c)}-\frac{b}{(a-b)(b-c)}+\frac{c}{(a-c)(b-c)}.$$

The L.C.M. of the denominators is $(a-b)(b-c)(a-c)$.
The expression may therefore be written

$$\frac{a(b-c)-b(a-c)+c(a-b)}{(a-b)(b-c)(a-c)}=0.$$

Ex 2. Simplify $\frac{x-2a}{x+a}+\frac{2(a^2-4ax)}{a^2-x^2}-\frac{3a}{x-a}$.

By changing the sign of the last fraction, the L.C.M. of the denominators becomes a^2-x^2. The expression may then be written

$$\frac{(x-2a)(a-x)+2a^2-8ax+3a(a+x)}{a^2-x^2}$$

$$=\frac{3a^2-2ax-x^2}{a^2-x^2}=\frac{3a+x}{a+x}.$$

When it is required to simplify an expression containing the algebraic sum of three or more given fractions, it is usually convenient to take the L.C.M of the denominators as a common denominator. But, if this course be always followed, much unnecessary labour will often result It is sometimes better first to arrange the terms in groups of two or more together and simplify each group before proceeding further.

Ex 3. Simplify $\frac{1}{x-2}-\frac{1}{x-3}+\frac{1}{x-4}-\frac{1}{x-5}$

Here, following the ordinary rule, the L.C.M. of the denominators would be $(x-2)(x-3)(x-4)(x-5)$, and each numerator would have to be multiplied by three factors. Instead, we may simplify the first two terms,

$$\frac{1}{x-2}-\frac{1}{(x-3)}=\frac{x-3-x+2}{(x-2)(x-3)}=\frac{-1}{(x-2)(x-3)}.$$

In a similar manner, the remaining two terms become

$$\frac{1}{x-4}-\frac{1}{(x-5)}=\frac{-1}{(x-4)(x-5)}.$$

Hence, the given expression is equivalent to

$$-\frac{1}{(x-2)(x-3)}-\frac{1}{(x-4)(x-5)}$$

$$=\frac{-(x^2-9x+20)-(x^2-5x+6)}{(x-2)(x-3)(x-4)(x-5)}$$

$$=\frac{-2(x^2-7x+13)}{(x-2)(x-3)(x-4)(x-5)}.$$

Fractions of the form $\frac{x^3+y^3}{x+y}$ *are easily simplified by writing down* **the factors of the numerator.** Thus

$$\frac{x^3+y^3}{x+y}=\frac{(x+y)(x^2-xy+y^2)}{x+y}$$
$$=x^2-xy+y^2.$$

Similarly, $\frac{x^4-y^4}{(x+y)(x-y)}=\frac{(x^2+y^2)(x^2-y^2)}{x^2-y^2}=x^2+y^2,$

and $\qquad x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2).$

The above examples are simple applications of the following general statements

Factors.

x^n+y^n is divisible by $x+y$ when n is odd;
x^n-y^n „ „ $x+y$ „ n is even;
x^n-y^n „ „ $x-y$ „ n is either odd or even.

Surd quantities.—In questions dealing with fractions involving surd quantities, simplification is often effected by using one or both of the forms $(a+b)^2=a^2+2ab+b^2$ (i) or $(a^2-b^2)=(a+b)(a-b)$ (ii).

The former may be used in extracting the root of a binomial surd quantity. Some applications are indicated in the following examples .

Ex. 1. Simplify (i) $\frac{1}{\sqrt{20}}$; (ii) $\frac{\sqrt{5}-2}{\sqrt{5}+2}$ and express the result in each case as a decimal fraction. (Given $\sqrt{5}=2\ 2361$.)

(i) Here $\frac{1}{\sqrt{20}}=\frac{\sqrt{20}}{20}=\frac{\sqrt{4\times5}}{2\times10}=\frac{\sqrt{5}}{10}.$

But $\qquad \sqrt{5}=2\cdot2361$; $\qquad \frac{\sqrt{5}}{10}=0\cdot22361.$

(ii) $\frac{\sqrt{5}-2}{\sqrt{5}+2}.$

Multiply numerator and denominator by $\sqrt{5}-2$.

$$\frac{\sqrt{5}-2}{\sqrt{5}+2}=\frac{(\sqrt{5}-2)(\sqrt{5}-2)}{(\sqrt{5}+2)(\sqrt{5}-2)}$$

Apply the form given by Eq. (ii) above, and

$$\therefore\ \frac{(\sqrt{5}-2)(\sqrt{5}-2)}{(\sqrt{5}+2)(\sqrt{5}-2)}=\frac{(\sqrt{5}-2)^2}{5-4}=9-4\sqrt{5}=0\cdot0556.$$

Ex. 2. Show without extracting roots that $\sqrt{17}+\sqrt{19}$ is less than $6\sqrt{2}$.

Here, if $\sqrt{17}+\sqrt{19}<6\sqrt{2}$,

then, squaring both sides,

$$17+2\sqrt{17\times 19}+19<72;$$

$$\therefore\ 36+2\sqrt{17\times 19}<72,$$

$$\text{or}\ 36+2\sqrt{323}<72.$$

Subtracting 36 from each side and dividing by 2 we obtain

$$\sqrt{323}<18.$$

Squaring both sides $323<324$.

Ex. 3. Find the value of

$$\frac{3-\sqrt{5}}{(\sqrt{3}+\sqrt{5})^2}+\frac{3+\sqrt{5}}{(\sqrt{3}-\sqrt{5})^2}.$$

As a common denominator take the product of the two denominators. Then

$$\frac{(3-\sqrt{5})(\sqrt{3}-\sqrt{5})^2+(3+\sqrt{5})(\sqrt{3}+\sqrt{5})^2}{(\sqrt{3}+\sqrt{5})^2\times(\sqrt{3}-\sqrt{5})^2}$$

or

$$\frac{(3-\sqrt{5})(3+5-2\sqrt{15})+(3+\sqrt{5})(3+5+2\sqrt{15})}{(3-5)^2}$$

$$=12+5\sqrt{3}=20{\cdot}66.$$

In Ex. 3, and in all similar cases, the numerical values of numerator and denominator may be obtained by using a table of square roots, then the value of each fraction may be obtained by logarithms

Ex. 4. In the expression $(x-a)^2-(y-b)^2$ put $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$, and $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$, and reduce the resulting expression to its simplest form.

$$(x-a)^2-(y-b)^2=(x-a+y-b)(x-a-y+b).$$

Substitute the given values for x and y, thus,

$$(x-a+y-b)=\left(a+b+\frac{(a-b)^2}{4(a+b)}-a+\frac{a+b}{4}+\frac{ab}{a+b}-b\right)$$

$$=\frac{(a-b)^2+(a+b)^2+4ab}{4(a+b)}=\frac{(a+b)^3}{2(a+b)}$$

$$=\frac{a+b}{2}.$$

Similarly, for the second factor,

$$(x-a-y+b)=\left\{a+b+\frac{(a-b)^2}{4(a+b)}-a-\left(\frac{a+b}{4}+\frac{ab}{a+b}\right)+b\right\}$$

$$=\frac{8b(a+b)+(a-b)^2-\{(a+b)^2+4ab\}}{4(a+b)}$$

$$=\frac{8b^2}{4(a+b)}=\frac{2b^2}{a+b}$$

Hence, $(x-a)^2+(y-b)^2$

$$=\frac{a+b}{2}\times\frac{2b^2}{a+b}$$

$$=b^2.$$

Ex. 5. If $x^2=(x+1)$, show that $x^5=5x+3$.

$x^5=x^4\times x=(x+1)^2\times x$ by substitution,

$x^5=(x^2+2x+1)x=(3x+2)x,$

$\therefore\ x^5=5x+3$

Partial fractions.—A single fraction has often to be expressed as the sum of several simpler fractions. Such fractions are called partial fractions

If necessary, the given fraction must be simplified, and it may be assumed that the denominator can be resolved into its factors The methods adopted in some easy cases may be seen from the following examples.

Ex. 1. Express in the form of partial fractions $\frac{2x+1}{x^2-5x+6}$.

The factors of the denominator are $(x-2)$ and $(x-3)$.

First write the given fraction in the form.

$$\frac{2x+1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}, \qquad \text{(1)}$$

where the values of the coefficients A and B are determined.

Multiplying both sides of Eq (1) by $(x-2)(x-3)$, we obtain

$$2x+1=A(x-3)+B(x-2). \qquad \text{(2)}$$

By putting in succession $x-3=0$ and $x-2=0$, the numerical values of A and B can be found.

Thus, let $x-3=0$ Then substitute $x=3$ in (2).

$$7=B(3-2);\ \therefore\ B=7$$

Again, let $x-2=0$; $\therefore\ x=2$

Then $5=A(2-3)$, or $A=-5$.

Substitute these values in (1), thus,

$$\frac{2x+1}{(x-2)(x-3)} = -\frac{5}{x-2} + \frac{7}{x-3}.$$

Another method which may be used is as follows:

$$\frac{2x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Multiply by $x-2$;

$$\therefore \frac{2x+1}{x-3} = A + \frac{B(x-2)}{x-3}.$$

Now put $x=2$

Then $\frac{5}{-1} = A + 0; \quad \therefore A = -5.$

Similarly, multiplying (1) by $x-3$ and putting $x=3$, we obtain

$$\frac{7}{1} = 0 + B; \quad \therefore B = 7.$$

Ex. 2. Express in partial fractions the fraction $\frac{2x+1}{x^3-6x^2+11x-6}$.

The denominator is $(x-1)(x-2)(x-3)$,

Let $\frac{2x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$

Multiply both sides by $(x-1)(x-2)(x-3)$;

$$\therefore 2x+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2). \quad \text{(3)}$$

Let $x-1=0; \quad x=1.$

Substitute this value for x, then from (3),

$$3 = A(1-2)(1-3) = 2A;$$

$$A = \frac{3}{2}$$

Similarly, let $x-2=0; \quad \therefore x=2.$

Substitute in (3); then

$$5 = B(2-1)(2-3) = -B;$$

$$\therefore B = -5$$

Finally, put $x-3=0$, or $x=3$;

$$\therefore 7 = 2C, \text{ or } C = \frac{7}{2}.$$

Hence, $\frac{2x+1}{(x-1)(x-2)(x-3)} = \frac{3}{2(x-1)} - \frac{5}{x-2} + \frac{7}{2(x-3)}.$

Ex. 3. Resolve into partial fractions the single fraction

$$\frac{lx^2+mx+n}{(x-a)(x-b)(x-c)}.$$

Let
$$\frac{lx^2+mx+n}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}.$$

Multiply throughout by $(x-a)(x-b)(x-c)$,

$$\therefore\ lx^2+mx+n=A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b).$$

Let the factor $x-a=0$, $\quad x=a$,

then,
$$la^2+ma+n=A(a-b)(a-c);$$

$$\therefore\ A=\frac{la^2+ma+n}{(a-b)(a-c)}$$

In a similar manner let $x-b=0$, $\therefore\ x=b$;

then
$$B=\frac{lb^2+mb+n}{(b-a)(b-c)}.$$

Finally, if $x-c=0$, we obtain

$$C=\frac{lc^2+mc+n}{(c-a)(c-b)}$$

If the numerator is of equal, or greater, degree than the denominator, it will be necessary to divide the former by the latter, so that the fraction to be operated upon shall have its numerator of lower degree than its denominator. Also, when the denominator of a fraction contains a factor such as $(x-a)^3$, it is necessary to take several corresponding partial fractions having for their denominators the factors $x-a$, $(x-a)^2$, $(x-a)^3$, etc

Ex 4. Resolve into partial fractions

$$\frac{3x+5}{(1-2x)^2}.$$

Let
$$\frac{3x+5}{(1-2x)^2}=\frac{A}{1-2x}+\frac{B}{(1-2x)^2}.$$

Multiply both sides by $(1-2x)^2$. Then

$$3x+5=A(1-2x)+B. \qquad (1)$$

Put $1-2x=0$, $\quad\therefore\ x=\frac{1}{2}$;

$$\therefore\ \frac{3}{2}+5=B,$$

giving
$$B=\frac{13}{2}.$$

Substitute this value for B in (1),

$$3x+5-\frac{13}{2}=A(1-2x);$$

$$\therefore\ A=-\frac{3}{2}\left(\frac{1-2x}{1-2x}\right)=-\frac{3}{2}.$$

Or, put $x=0$ in (1), then $5=A+\frac{13}{2}$; $\therefore\ A=-\frac{3}{2}$.

Hence, $$\frac{3x+5}{(1-2x)^2}=\frac{13}{2(1-2x)^2}-\frac{3}{2(1-2x)}.$$

A very useful artifice which may be used in many cases (especially in dealing with factors such as $(x-a)^n$ and often referred to as **repeating factors**) may be shown by an example

Ex 5. Resolve into partial fractions

$$\frac{x^3+3x+1}{(1-x)^4}. \qquad \ldots\ldots\ \ldots\ldots\ \ldots\ldots(1)$$

Let $1-x=z$; $\therefore\ x=1-z$. Substitute in Eq. (1) and we obtain

$$\frac{(1-z)^3+3(1-z)+1}{z^4}$$

$$=\frac{1-3z+3z^2-z^3+3-3z+1}{z^4}=\frac{5-6z+3z^2-z^3}{z^4}$$

$$=\frac{5}{z^4}-\frac{6}{z^3}+\frac{3}{z^2}-\frac{1}{z}.$$

Then, substituting for z, this result may be written,

$$\frac{5}{(1-x)^4}-\frac{6}{(1-x)^3}+\frac{3}{(1-x)^2}-\frac{1}{1-x}.$$

Thus, in Ex. 4 let $1-2x=z$, $\therefore\ x=\frac{1-z}{2}$;

$$\therefore\ \frac{3x+5}{(1-2x)^2}=\frac{\frac{3}{2}(1-z)+5}{z^2}=\frac{13-3z}{2z^2}$$

$$=\frac{13}{2z^2}-\frac{3}{2z};$$

$$\therefore\ \frac{3x+5}{(1-2x)^2}=\frac{13}{2(1-2x)^2}-\frac{3}{2(1-2x)}.$$

EXERCISES. I.

1. Simplify $$\left(\frac{x^5-1}{x-1}\right)^2-\left(\frac{x^5+1}{x+1}\right)^2$$
and find its numerical value when $x\sqrt{(2+\sqrt{3})}=1$.

2. Find the value of $\sqrt{\left(\frac{\sqrt{5}-2}{\sqrt{5}+2}\right)}$ to three places of decimals.

3. Find the product of $\frac{a}{4}+\frac{\sqrt{ab}}{3}+\frac{b}{9}$ and $\frac{\sqrt{a}}{2}-\frac{\sqrt{b}}{3}$, and find the value of the product when $a=12$ and $b=18$.

4. Simplify $\frac{x^2-8x+12}{3x^2-17x-6}-\frac{2x^2+5x+2}{6x^2+x-1}$, and find its value when $3x=\sqrt{2}-1$.

5. Reduce to its simplest form:
$$\frac{x^3-5x^2-8x+12}{x^4-7x^3+7x^2-7x+6}.$$
Find its value when $x=1+\sqrt{3}$ $(\sqrt{3}=1{\cdot}732.)$

Simplify the following expressions:

6. $\sqrt{(52-7\sqrt{12})}$.

7. $\frac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}-\frac{1-\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$.

8. $\frac{4\sqrt{2}-3\sqrt{3}}{7-2\sqrt{6}}\times\frac{2\sqrt{2}+\sqrt{3}}{7-2\sqrt{2}}$.

9. If $$\left(\frac{1}{x}+\frac{2}{y}+\frac{1}{z}\right)=\frac{(x+2y+z)^2}{xy^2z}$$
show that either $x=z$ or $y^2=zx$.

10. Show that $\left(x-2+\frac{1}{x}\right)\left(x+2+\frac{1}{x}\right)\left(x^2+2+\frac{1}{x^2}\right)=\left(x^2-\frac{1}{x^2}\right)^2$.

11. Given $\sqrt{5}=2{\cdot}236$, express $\frac{1}{\sqrt{20}}$ and $\frac{\sqrt{5}-2}{\sqrt{5}+2}$ as decimals.

12. Find the value of $\left(x+\frac{a}{b}\right)\left(x+\frac{b}{a}\right)-\left(x-\frac{a}{b}\right)\left(x-\frac{b}{a}\right)$ when $x=\frac{1}{a^2+b^2}$.

13. Reduce the following expression to its simplest form: $\left(\frac{a-b}{a+b}\right)-\left(\frac{a}{a-b}+\frac{b}{b-a}\right)^2$; and find its value, expressed as a decimal, when $a=2$ and $b=\sqrt{5}=2{\cdot}236$.

14. Simplify $(a+b+c)^3+6a(a-b-c)(a+b+c)+(a-b-c)^3$.

15. Simplify $\dfrac{1+2\sqrt{x}}{1-\sqrt{x}}-\dfrac{1-\sqrt{x}}{1+2\sqrt{x}}$; and find its value to four places of decimals when $3x=1$, having given $\sqrt{3}=1\cdot732$.

16. If $a^2=m+n$, $b^2=n+l$, $c^2=l+m$ and $2s=a+b+c$ show that

$$s(s-a)(s-b)(s-c)=\tfrac{1}{4}(mn+nl+lm).$$

17. Simplify $\dfrac{x^2-x-2}{x^2-3x+2}+\dfrac{2x^2+x-3}{2x^2+5x+3}-2$; and find its value to four places of decimals, when $x=1+\sqrt{3}$.

Simplify the expressions:

18. $\dfrac{[ax^2+(b-c)x+d]^2-[ax^2+(b+c)x+d]^2}{[ax^2+(b+e)x+d]^2-[ax^2+(b-e)x+d]^2}$.

19. $\dfrac{(x+y)^2+2(x^2-y^2)+(x-y)^2}{(x^4-2x^2y^2+y^4)\left\{\dfrac{1}{(x-y)^2}+\dfrac{2}{x^2-y^2}+\dfrac{1}{(x+y)^2}\right\}}$.

Resolve into factors:

20 $12x^2-25xy+12y^2$.

21. $a^8+a^4b^4+b^8$.

22 $x^4+x^2y^2+y^4-2xy-1$.

23. Show that a is a factor of the expression

$$(a+b)^2(a^2+c^2)-(a+c)^2(a^2+b^2).$$

Resolve into factors the following expressions:

24. $20x^2-x-30$

25 $2xy+7x+6y+21$.

26 $5x^2-(7+15a)x+21a$.

27 $x^4-1-4(x-1)$.

Simplify the following expressions:

28 $$\left(a-\frac{a-b}{1+ab}\right)\times\frac{a}{b}\div\left(1+\frac{a(a-b)}{1+ab}\right)$$

29. $\dfrac{x^2-x}{x^2-1}\times\dfrac{(x+1)^2-(x-1)^2}{2x}-\left(\dfrac{x}{x+1}-1\right)+\left(\dfrac{x^3-1}{x^2-1}-1\right)$

30. Given $\sqrt{2}=1\cdot4142$, and $\sqrt{3}=1\cdot7321$, find the value of $\dfrac{1}{\sqrt{6}-\sqrt{2}}$ correct to three places of decimals, using a contracted method of multiplication.

31. Find the value of

$$\frac{3-\sqrt{5}}{(3+\sqrt{5})^2}+\frac{3+\sqrt{5}}{(3-\sqrt{5})^2}.$$

32. If $z=\sqrt{(x^2+y^2)}$ show that

$$\frac{x+y+z}{-x+y+z}=\frac{x-y+z}{x+y-z}.$$

33. Show that

$$\frac{z^2}{a^2+b^2}+\frac{a^2+b^2}{a^2b^2}\left(x-\frac{za^2}{a^2+b^2}\right)^2=\frac{x^2}{a^2}+\left(\frac{z-x}{b}\right)^2.$$

Express in partial fractions:

34. $\dfrac{2x-5}{(x-2)(x-3)}$.

35. $\dfrac{7x-1}{1-5x+6x^2}$.

36. $\dfrac{9}{(x-1)(x+2)^2}$.

37. $\dfrac{x-13}{x^2-2x-15}$.

38. $\dfrac{x-5}{x^2-x-2}$.

39. $\dfrac{x+37}{x^2+4x-21}$.

40. $\dfrac{5x-18}{x^2-7x+12}$

41. $\dfrac{3x^2-10x-4}{(x-2)(x-4)}$.

42. $\dfrac{2x^3-11x^2+12x+1}{(x-1)(x-2)(x-3)}$

43. $\dfrac{5+2x-3x^2}{(x^2-1)(x+1)}$.

CHAPTER II.

MEASUREMENT OF ANGLES AND THE SIMPLE RATIOS.

Measurement of angles.—In the measurement of length, a certain distance is selected as a unit, and the number of times a given length contains the unit length is the measure of its length. In like manner, the magnitude of an angle is estimated by the number of times it contains the unit angle. The two angular units adopted are the **degree** and the **radian**.

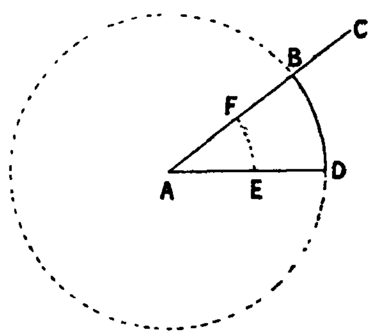

Fig 1

Let AE be a line free to move about a centre A. Any point in the line such as D (Fig. 1) will eventually describe a circle If we assume such a circle to be divided into 360 equal parts then the lines joining any two consecutive points on the circumference to the centre A will enclose an angle of one degree, which is written 1°.

A degree is divided into 60 minutes and a minute into 60 seconds An angle of thirty degrees, twenty minutes, and fifteen seconds would be written $30^\circ\ 20'\ 15''$.

The actual distance described by B will be proportional to the amount of turning from the initial position, also for the same angle the arc described is proportional to the radius, hence the measure of an angle is denoted by $k\dfrac{\text{arc}}{\text{radius}}$; where k

is a constant whose value depends on the particular system adopted. Thus $k=1$ in the *radian* system, and $k=180\div$ratio *of circumference to diameter, in the degree* system.

Assume AB, Fig. 1, a line initially coincident with the line AD, to be rotated about a centre A into the position AB, through an angle which may be denoted by θ.

To ascertain the magnitude of the angle, draw with A as centre an arc of a circle cutting AD in E and AB in F. Then the ratio $\dfrac{\text{arc}}{\text{radius}}$ is called the measure of the angle in radians,

$$\therefore \ \textbf{angle in radians} = \frac{\textbf{arc}}{\textbf{radius}} \ \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The measure of the angle will obviously be unity when the numerator is equal to the denominator, or when the length of arc DB is equal to the radius AD.

The unit angle is called a **radian**, and its value is $\dfrac{180^\circ}{\pi}$, or is equal to $57^\circ\, 17'\, 45''$ nearly, or about $57^\circ\, 3$.

Hence, to convert to radians an angle given in degrees, it is necessary to divide by 57 3. Similarly, to convert an angle from radians to degrees, multiply by 57 3

From (i) we have **angle × radius = arc**

Hence, when any two of the three terms are given the remaining term may be obtained.

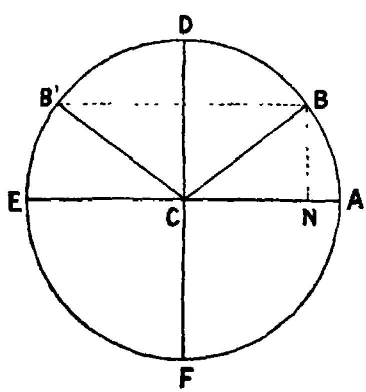

FIG. 2.—Ratios of angles

Ratios of angles.—The ratios of an angle designated as **sine**, **cosine**, and **tangent**, abbreviated into sin, cos, and tan, are probably already familiar to the reader. It is only necessary to refer briefly to the definitions.

When the rotating line (Fig 2) moving in a direction opposite to the hands of a clock comes into the position CB, then, if BN be drawn perpendicular to CN and meeting CA in N, and the angle NCB

be represented by θ, we have for the triangle the following relations .

$$\sin\theta=\frac{BN}{CB},\quad \cos\theta=\frac{CN}{CB},\quad \tan\theta=\frac{BN}{CN}$$

Also $\sin^2\theta+\cos^2\theta=1$, since $BN^2+CN^2=CB^2$.

The reciprocals of each of these ratios are also important and are as follows

$$\text{cosecant }\theta=\frac{1}{\sin\theta}=\frac{CB}{BN},\quad \text{secant }\theta=\frac{1}{\cos\theta}=\frac{CB}{CN},$$

$$\text{cotangent }\theta=\frac{1}{\tan\theta}=\frac{CN}{BN}.$$

The abbreviations cosec θ, sec θ, and cot θ, are used for these ratios. Also, referring to Fig. 1, it is easily seen that

$$\sec^2\theta=1+\tan^2\theta.$$

The ratios of the sine, cosine, and tangent for 30°, 45°, and 60° are very important, and are so often required in calculations that it is necessary to remember their numerical values

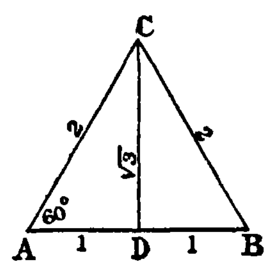

Fig. 3.—Angles of 30° and 60°.

Ratios for 60°, 30°.—One of the best methods is to draw (or better, mentally to picture) an equilateral triangle ABC (Fig. 3), with each of its sides say 2 units length. If from the vertex C a perpendicular CD be drawn to the opposite side, then, as ADC is a right-angled triangle, the length of CD is

$$\sqrt{2^2-1^2}=\sqrt{3}.$$

Hence $\sin A=\sin 60°=\frac{\sqrt{3}}{2},\quad \cos 60°=\frac{1}{2},\quad \tan 60°=\sqrt{3}.$

The angle ACD is an angle of 30°. Hence we get the ratios

$$\sin 30°=\frac{1}{2},\quad \cos 30°=\frac{\sqrt{3}}{2},\quad \tan 30°=\frac{1}{\sqrt{3}}.$$

Ratios for 45°.—Draw a right-angled triangle in which one side AB is equal to the other side BC. Then ABC is an

isosceles triangle and the angles at A and C are in each case 45°. If in Fig. 4 the lengths of the sides AB and BC be denoted by 1, then

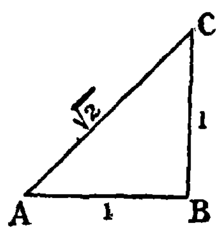

FIG 4.—Angle of 45°

$$AC = \sqrt{2}.$$

Hence $$\sin 45^\circ = \frac{1}{\sqrt{2}},$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}},$$

$$\tan 45^\circ = 1.$$

Complementary angles.—Two angles are said to be complementary when their sum is 90° (a right angle).

Ex. Let $A = 30^\circ$, $B = 60^\circ$, then, as we have found above,

$$\sin A = \cos B, \quad \text{and} \quad \cos A = \sin B;$$

these relations hold generally, and we have

$$\begin{aligned} \sin A &= \cos(90^\circ - A), \\ \cos A &= \sin(90^\circ - A), \\ \tan A &= \cot(90^\circ - A), \\ \cot A &= \tan(90^\circ - A), \\ \sec A &= \operatorname{cosec}(90^\circ - A), \\ \operatorname{cosec} A &= \sec(90^\circ - A). \end{aligned}$$

Angles greater than 90°.—The ratios of the sine, cosine, tangent, etc., which are all positive for angles not exceeding 90°, may or may not be positive for angles greater than 90°.

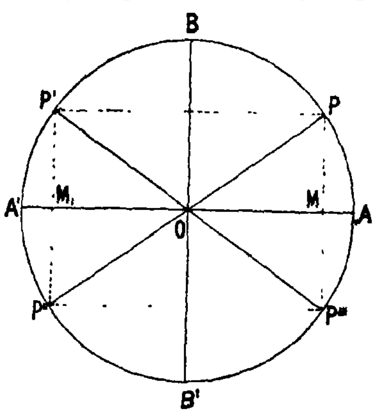

FIG 5.—Ratios of angles greater than 90°

The conventions adopted are as follows If a circle be drawn as in Fig. 5 and also horizontal and vertical diameters, as AA', BB', then all distances measured to the right of the line BB' are said to be **positive**, and those to the left are said to be **negative**.

Distances measured upwards from AA' are positive, those

downwards are negative. The revolving line itself is always positive, but angles are reckoned positive or negative according as the revolving line rotates in the opposite or the same direction as the hands of a watch Thus, if AP be one-twelfth of the circumference, then, joining P to O, the angle POA is an angle of 30°. If $M_1P'=MP$ the angle AOP' is 150°, and

$$\sin 150^\circ = \frac{M_1P'}{OP'} = \frac{MP}{OP} = \frac{1}{2}.$$

The perpendicular M_1P' is measured in a positive direction; OM_1 is measured in a negative direction;

$$\cos 150^\circ = \frac{OM_1}{OP'} = -\frac{\sqrt{3}}{2}.$$

In a similar manner, if $A'OP''$ is an angle of $180^\circ+30^\circ=210^\circ$, both sine and cosine are negative. Finally, corresponding to the position P''', the sine of the angle is negative and the cosine is positive

As the tangent is the ratio of sine to cosine, it follows that when the sine and cosine have the same sign, either positive or negative, the tangent is positive, but is negative when the sine and cosine have different signs. Some values are given in the following table; these should be carefully verified.

Collecting the results for the points P, P', P'', and P''' we find

Angle	30°	150°	210°	330°
sin	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
cos	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
tan	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

General values.—It has been seen that an angle is traced out by the revolution of a line, from coincidence with another line into a second position; and, as the angle may be traced

out by any number of revolutions of the line, it follows that for a given value of a trigonometrical ratio there is an indefinite number of angles. But corresponding to a given angle there is only one value for each ratio.

If n is used to denote any integer, $2n$ represents an **even** number, and $2n+1$ or $2n-1$ an **odd** number; positive and negative values may be ensured by using the symbol $(-1)^n$.

$(-1)^n$ is $+1$ when n is even including zero, and is -1 when n is odd.

To find a general expression for all the angles which have a given sine or cosecant—

Let CP, a line initially coincident with CB, move into a position CP, so that the angle BCP is θ; if CP_1 is another position of CP so that $P_1B_1 = PB$, the two angles BCP and B_1CP_1 are equal, and $\sin\theta = \sin(180° - \theta)$

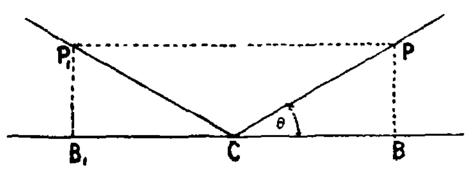

FIG 6

These angles may be increased by any number of revolutions of the line CP, that is by any multiple of four right angles, or $2n\pi$. It will then be obvious that all angles having the same sine, or cosecant, are included in the formulae

$$n\pi + (-1)^n\theta$$

In a similar manner, all the angles which have a given cosine, or secant, are included in the formulae

$$2n\pi \pm \theta.$$

And all the angles which have a given tangent, or cotangent, are included in the general formula

$$n\pi + \theta.$$

Graphical measurement of angles.—In graphical work in which angles occur, the magnitudes should be set out, or measured, as accurately as possible. Thus, when two sides and the included angle of a triangle are given, the two sides may be marked off as accurately as a good scale will permit,

but the results obtained will be inaccurate if an error is made in setting out the given angle.

The usual method adopted in setting out a given angle is to use some form of **protractor**. These are made both in the form of a rectangle and of a semicircle, but are rarely sufficiently accurate to enable the results obtained by them to be more than a check on calculated values. The most accurate results are probably obtained by using a good scale, a pair of compasses, and either a table of chords of angles or a table of tangents, Table VI.

Table of chords.—To set out a given angle at A (Fig. 7), make the base AB equal, on any convenient scale, to, say, 10 units; with A as centre and AB as radius, describe an arc. In Table VIII corresponding to the given angle, a number of

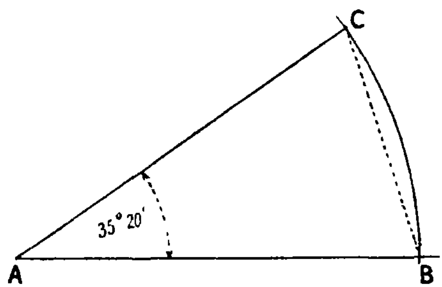

FIG 7.—Use of a table of chords

three figures is tabulated Multiply the number obtained from this table by 10, and, with B as centre, and the length so obtained as radius, describe an arc intersecting the former in C, join A to C Then BAC is the angle required.

Ex. 1. Set out at a given point, A, an angle of 35° 20′

Measure off AB equal to 10 inches, and describe an arc with A as centre and AB as radius. Opposite the angle 35° 20′ in Table VIII the value 0·607 is tabulated. Multiplying this by 10 we obtain 6·07. With B as centre and a radius 6 07, describe an arc BC intersecting the former in C. Join A to C. Then BAC is an angle of 35° 20′.

The converse of this exercise, *i.e.* given an angle to obtain its measure, will not present much difficulty Either of the lines meeting at the vertex of the angle may be assumed as

base and a length of 10 units marked off. Then, with this distance as radius, an arc of a circle may be drawn cutting both the lines enclosing the angle. The chord can be measured and divided by 10, finally by referring to Table VIII. the numerical measure of the angle is ascertained.

Table of tangents.—An angle can be determined graphically when the numerical value of its tangent is known.

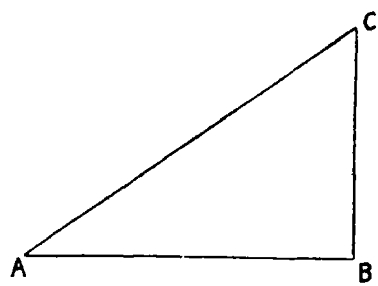

FIG 8 —Use of a table of tangents

Ex. 2. Set out an angle of 35° 20′. Make AB (Fig 8) equal to (say) 10 units and draw BC perpendicular to AB In Table VI., corresponding to 35° 20′, the value 0 7089 is tabulated Multiply this value by 10 and make BC equal to 7·089 Join A to C. Then BAC is an angle of 35° 20′.

EXERCISES II.

1. Express seven-sixteenths of a right-angle in radians.

2 What is meant by the radian measure of an angle? How many degrees and minutes are there in an angle whose radian measure is $\frac{5}{6}$?

3. Express in radians an angle of 240° and express in degrees the angle $\frac{2\pi}{3}$ (radians).

4. The difference of two angles is 10°, the radian measure of their sum is 2; find the radian measure of each angle.

5. Find the distance in miles between two places on the Equator which differ in longitude by 6° 18′, assuming the Earth's equatorial diameter to be 7926 miles.

6. What is the unit of radian measure? Find the length of that part of a circular railway curve which subtends an angle of $22\frac{1}{2}^\circ$ to a radius of a mile.

7. Write down the values of $\sin 132^\circ$, $\cos 226^\circ$, $\tan 326^\circ$.

8. Write down the values of $\sin 165^\circ$, $\cos 132^\circ$, $\tan 198^\circ$.

9. Write in a table the values of the sine, cosine, and tangent of the following angles, 23°, 123°, 233°, 312°, 383°.

Find the measure in radians of an angle of 384°.

10. Trace the variations in sign and magnitude of $\cos A - \sin A$, as A varies from 0° to 180°.

11. Find the two least values of θ if $\sin\theta = \sqrt[3]{\frac{a}{b}}$ where $a = 2\cdot12$, $b = 6\ 47$

12. The geographical mile being a minute of latitude on the surface of the Earth, supposed spherical, prove that the circumference of the Earth is 21600 geographical miles.

CHAPTER III.

RATIOS OF THE SUM AND DIFFERENCE OF ANGLES.

Trigonometrical ratios.—In considering trigonometrical ratios, it should be carefully borne in mind that in all except the simplest case of the acute angle, it is of the utmost importance to be quite clear in regard to the direction in which the various lines are drawn. When this is made out, there will be no difficulty in dealing with angles of any magnitude.

Any angle such as XAP (Fig 9) traced by a line AP, initially coincident with a fixed line AX, and rotating about a fixed point A in the opposite direction to the hands of a clock (or anti-clockwise), may, as has been seen, be expressed numerically by the number of *degrees* or *radians* in the angle, or simply be indicated by a letter, such as A

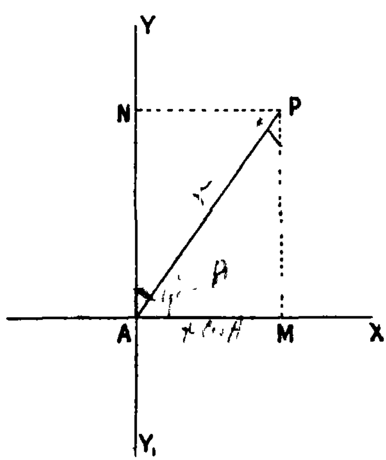

FIG 9.—Projection of a line.

Such a line as AP carries with it a number of associated lines, or ratios, and although these are probably familiar to the reader, it may be useful to refer briefly to them here, and especially to indicate how, by means of such ratios, angles of any magnitude may be represented.

If from P, a line PM be drawn perpendicular to AX and meeting AX in M, and similarly PN is drawn perpendicular

to AY, then AM is called the **projection** of AP on AX; and AN the projection on AY. The following ratios are at once obtained:

$$\frac{MP}{AP}=\sin A, \quad \frac{AM}{AP}=\cos A, \quad \frac{MP}{AM}=\tan A,$$

if $AP=r$, then the projection $AM=r\cos A$; or, **the projection of a line of length r on another to which it is inclined at an angle A is $r\cos A$.**

Since AP may denote the edge view of an area, the preceding statement may be applied to an area.

The angle $APM=NAP$ (Fig. 9),

$$\sin A=\frac{PM}{AP}=\frac{AN}{AP}$$

But $$\frac{AN}{AP}=\cos NAP=\cos(90^\circ-A)=\sin A.$$

Hence, the projection of a vector r on an axis AX to which it is inclined at an angle A, is $r\cos A$; and on the axis OY, or axis of y, is $r\sin A$. The two projections just referred to are called the **rectangular components of the vector AP.**

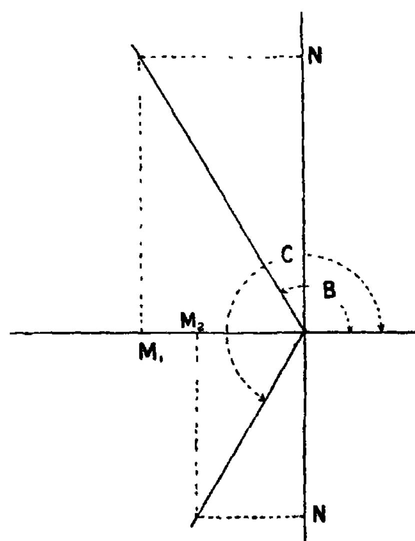

FIG 10 —Rectangular components.

In the case of an obtuse angle B (Fig 10), the projection is in the negative direction, and the cosine is negative. The sine remains positive Thus, if B is 120°,

$$\cos 120^\circ=-\cos 60^\circ;$$
$$\sin 120^\circ=\sin 60^\circ,$$
$$\tan 120^\circ=-\tan 60^\circ.$$

For an angle between 180° and 270°, say the angle C, the projections giving the sine and cosine of C are both negative, while the tangent is positive.

Finally, for an angle between 270° and 360°, it will easily be made out from its projections that the sine is negative, the cosine is positive, and the tangent is therefore negative.

Negative angles.—As already indicated, positive angles are angles formed by the rotation of a line in the opposite direction to the hands of a clock It is, however, sometimes convenient to deal with angles formed by a line rotating in the opposite direction, or clockwise. Such angles are called **negative angles.** Thus, an angle of 340° could be obtained by the rotating line describing an angle of 340° in a positive direction, or an angle of 20° in a negative direction.

The ratios for such angles (Fig 11) are found by the same rule as for positive angles.

$$\text{Thus } \cos(-A)=\frac{OM}{OP} \text{ and is positive,}$$

$$\sin(-A)=\frac{ON_1}{OP} \text{ and is negative,}$$

$$\tan(-A)=\frac{ON_1}{OM} \text{ and is negative}$$

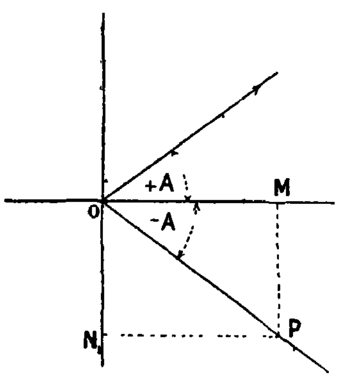

FIG 11 —Negative angles

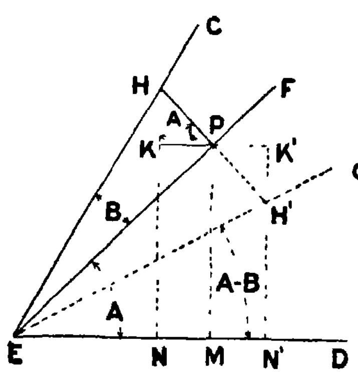

FIG 12 —Sum and difference of angles

Sum or Difference of two angles.—Let DEF (Fig. 12) denote an angle A, and FEC an angle B. At any point P in EF, draw PH at right angles to EF, meeting EC in H.

Draw HN and PM perpendicular to DE, and PK parallel to DE.

As the angle KPE is equal to A, and KPH is complementary to KHP and to KPE, it follows that the angle KHP is equal to A.

We have
$$\sin(A+B)=\frac{NH}{EH}=\frac{NK+KH}{EH}=\frac{MP+KH}{EH}$$
$$=\frac{MP}{EP}\cdot\frac{EP}{EH}+\frac{KH}{HP}\cdot\frac{HP}{EH}$$
$$=\sin A\cos B+\cos A\sin B;$$

similarly,
$$\cos(A+B)=\frac{EN}{EH}=\frac{EM-NM}{EH}=\frac{EM-KP}{EH}$$
$$=\frac{EM}{EP}\cdot\frac{EP}{EH}-\frac{KP}{PH}\cdot\frac{PH}{EH}$$
$$=\cos A\cos B-\sin A\sin B.$$

If the angle FEG is equal to B, then the angle DEG is $A-B$.

$$\sin(A-B)=\frac{N'H'}{EH'}=\frac{N'K'-H'K'}{EH'}$$
$$=\frac{MP-H'K'}{EH'}$$
$$=\frac{MP}{EP}\cdot\frac{EP}{EH'}-\frac{HK'}{PH'}\cdot\frac{PH'}{EH'}$$
$$=\sin A\cos B-\cos A\sin B.$$

The result may also be obtained by writing $-B$ for B in the preceding

In a similar manner,
$$\cos(A-B)=\cos A\cos B\cancel{+\sin A\sin B}.$$

So, too,
$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$
$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

By dividing numerator and denominator by $\cos A \cos B$, we obtain

$$\frac{\dfrac{\sin A \cos B}{\cos A \cos B}+\dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B}-\dfrac{\sin A \sin B}{\cos A \cos B}}=\frac{\tan A+\tan B}{1-\tan A \tan B}.$$

The last result may also be obtained geometrically as follows —

$$\tan(A+B)=\frac{NH}{EN}=\frac{NK+KH}{EM-NM}=\frac{MP+KH}{EM-KP}$$

Then, by dividing numerator and denominator by EM,

$$=\frac{\dfrac{MP}{EM}+\dfrac{KH}{EM}}{1-\dfrac{KP}{KH}\dfrac{KH}{EM}}.$$

But from the similar triangles PHK and PEM,

$$\frac{KH}{HP}=\frac{EM}{EP} \text{ or } \frac{KH}{EM}=\frac{HP}{EP},$$

$$\therefore \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A \tan B}.$$

By proceeding in a similar manner, we find

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A \tan B}.$$

Tests of the above formulae should be worked out by the student, using simple ratios, and the results obtained checked by reference to Table VI

Thus, if $A=45°$, $B=30°$, then $A+B=75°$

$$\tan(A+B)=\frac{\tan 45°+\tan 30°}{1-\tan 45° \tan 30°},$$

$$=\frac{1+\dfrac{1}{\sqrt{3}}}{1-\dfrac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}.$$

Thus, $\tan 75° = 3{\cdot}7321$, and referring to Table VI. opposite $\tan 75°$ we find this value tabulated.

Again, $A - B = 15°$,

$$\therefore \tan(A - B) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°},$$

$$= \frac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = 2 - \sqrt{3};$$

$$\therefore \tan 15° = 0{\cdot}2679,$$

and this is the value found in Table VI.

We have now found the following relations connecting simple with compound angles.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B, \qquad \ldots (1)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B, \qquad \ldots (2)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \qquad \ldots (3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \qquad \ldots (4)$$

These results may be combined thus,

$$\mathbf{\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B},$$

$$\mathbf{\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B},$$

$$\mathbf{\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}} \qquad \ldots (5)$$

By adding (1) and (2) we obtain,

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B \qquad \ldots (6)$$

We may conveniently replace $A + B$ by P, and $A - B$ by Q.

$$\therefore A + B = P,$$

$$A - B = Q,$$

or,

$$2A = P + Q, \quad \therefore A = \frac{P + Q}{2},$$

$$2B = P - Q, \quad \therefore B = \frac{P - Q}{2}.$$

Hence, by the appropriate modification of formulae (1) to (4), we obtain

$$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2},$$

$$\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2},$$

$$\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2},$$

$$\cos Q-\cos P=2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

These results may be expressed in words.

sum of two sines = twice the sine of half sum × cosine of half difference of the angles;

difference of two sines = twice the cosine of half sum × sine of half difference of the angles;

sum of two cosines = twice the cosine of half sum × cosine of half difference of the angles;

difference of two cosines = minus twice the sine of half sum × sine of half difference.

Formulae connecting an angle and the double angle.—If in the preceding formulae A is equal to B, then

$$\sin 2A=2\sin A\cos A,$$

$$\cos 2A=\cos^2 A-\sin^2 A$$

$$=2\cos^2 A-1=1-2\sin^2 A,$$

and $$\tan 2A=\frac{2\tan A}{1-\tan^2 A}.$$

We may replace $2A$ by A, if we also replace A by $\frac{A}{2}$;

$$\therefore\ \sin A=2\sin\frac{A}{2}\cos\frac{A}{2},$$

$$\cos A=2\cos^2\frac{A}{2}-1=1-2\sin^2\frac{A}{2}.$$

The preceding results may also be obtained in a more direct manner as follows:

Let NOP (Fig 13) be the angle A, and NOQ be the angle B. Draw the line OR bisecting the angle POQ.

Then, angle NOR
$=B+\frac{1}{2}(A-B)$
$=\frac{1}{2}(A+B)$.

Draw PRQ perpendicular to OR.

From points P, R, Q draw the perpendiculars PM, RL, and QN.

Then $ML=LN$.

Sum of the projections of OP and OQ on $OX=$ 2 (projection of OR), or

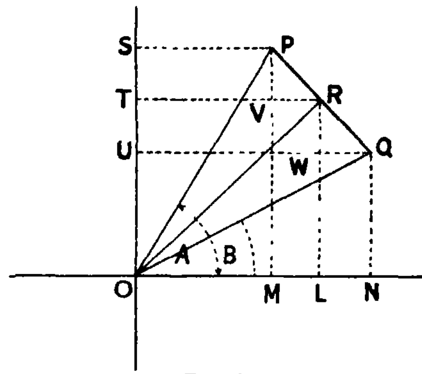

FIG 13.

$$OP\cos A+OQ\cos B=2OR\cos\tfrac{1}{2}(A+B) \quad \ldots\ldots\ldots(1)$$

Also $\quad OR=OP\cos POR=OP\cos\frac{1}{2}(A-B)$

Substituting this value of OR in (1)

$$OP\cos A+OQ\cos B=2OP\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$

As ORP and ORQ are equal and similar triangles, $OP=OQ$. Hence, dividing both sides by OP;

$$\therefore \cos \mathbf{A}+\cos \mathbf{B}=2\cos\tfrac{1}{2}(\mathbf{A}+\mathbf{B})\cos\tfrac{1}{2}(\mathbf{A}-\mathbf{B}).$$

By projecting on the axis OY we can obtain the sum of two sines

Thus, the projections of OP and OQ on OY is twice the projection of OR on OY;

$$OS+OU=2\times OT,$$

or $\quad OP\sin A+OQ\sin B=2\times OR\sin\frac{1}{2}(A+B)$,

but $\quad OR=OP\cos POR=OP\cos\frac{1}{2}(A-B)$,

$$OP\sin A+OQ\sin B=2\times OP\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B);$$

$$\therefore \sin \mathbf{A}+\sin \mathbf{B}=2\sin\tfrac{1}{2}(\mathbf{A}+\mathbf{B})\cos\tfrac{1}{2}(\mathbf{A}-\mathbf{B}).$$

From Fig. 13 it is seen that;

Projection of OQ on OX=projection of OP on OX together with projection of PQ on OX;

$$\text{projection of } OQ \text{ on } OX = OQ\cos B;$$
$$\text{,, } \quad OP \text{ on } OX = OP\cos A;$$
$$PQ = 2PR \text{ and } PR = OP\sin\tfrac{1}{2}(A-B),$$
$$\text{projection of } PR \text{ on } OX \text{ is } ML = RV = PR\sin\tfrac{1}{2}(A+B);$$
$$\text{also projection of } PQ = 2 \text{ projection of } PR$$
$$= 2PR\sin\tfrac{1}{2}(A+B).$$

Substituting for PR,

$$OQ\cos B = OP\cos A + 2OP\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B),$$

or $$OP(\cos B - \cos A) = 2OP\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B);$$

$$\mathbf{\cos B - \cos A = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).}$$

In the formulae for $\sin(A+B)$ and $\cos(A+B)$, by writing $-B$ for B we can obtain the corresponding formulae for $\sin(A-B)$ and $\cos(A-B)$.

Again, in $\sin(A-B)$, let $B=A$, then

$$\sin(A-A) = \sin 0 = \sin A\cos A - \cos A\sin A = 0.$$

In $\cos(A-B)$ let $B=A$,

$$\cos(A-A) = \cos 0 = \cos A\cos A + \sin A\sin A$$
$$= \cos^2 A + \sin^2 A = 1.$$

Inverse Ratios.—A very convenient method of writing $\sin\theta = \frac{5}{7}$ is to write it in the form $\theta = \sin^{-1}\frac{5}{7}$ which is read as the angle, the sine of which is $\frac{5}{7}$, this is also sometimes written arc $\sin\frac{5}{7}$. Thus, if $\sin\theta = 0{\cdot}4848$, this may be written either as $\theta = \sin^{-1}0{\cdot}4848$ or arc sin 0·4848. Similarly $\tan y = 0{\cdot}364$ may be written $y = \tan^{-1}0{\cdot}364$ or $y = \text{arc}\tan 0{\cdot}364$.

Numerical values.—We may use the formulae now obtained to find the numerical value of $\sin 15^\circ$, $\cos 75^\circ$, $\sin 75^\circ$, $\cos 15^\circ$, etc.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ$$
$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

As $\cos 75° = \sin 15°$ *this result is also the value of* $\cos 75°$. Or, we may proceed to find the value of $\cos 75°$ as follows:

$$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \text{ as before.}$$

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}},$$

and hence $\sin 75° = \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

The two fractions $\dfrac{\sqrt{3} \pm 1}{2\sqrt{2}}$ may be simplified in the usual way.

Thus, $\dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{2}(\sqrt{3}-1)}{4} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$.

The values of $\sqrt{6}$ and $\sqrt{2}$ can be at once obtained by logarithms or from a table of square roots;

$$\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{1 \cdot 0352}{4} = 0 \cdot 2588.$$

Referring to Table IV. opposite $\sin 15°$ we find this value tabulated

In a similar manner we have

$$\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = 0 \cdot 9659,$$

and this agrees with the value tabulated. Proceeding in this manner the student can make exercises for himself, taking various numerical data from Table IV., then obtain the sine, cosine, or tangent of the sum or difference of any two angles.

Thus, if $A = 20°$ and $B = 43°$.

Then $\sin(A + B) = \sin(20° + 43°)$

$$= \sin 20° \cos 43° + \cos 20° \sin 43°$$

$$= 0 \cdot 3420 \times 0 \cdot 7314 + 0 \cdot 9397 \times 0 \cdot 6820$$

$$= 0 \cdot 2501 + 0 \cdot 6409 = 0 \cdot 8910.$$

Referring to Table IV. we find that this value corresponds to $\sin 63°$;

$$\sin(20° + 43°) = \sin 63°.$$

From the formula

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

we have (when $A = B$)

$$\begin{aligned}\sin 2A &= \sin A \cos A + \cos A \sin A\\ &= 2 \sin A \cos A.\end{aligned}$$

Similarly, $\quad \cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$

We can, in like manner, proceed to find the values of $\sin 3A$ and $\cos 3A$

Thus,
$$\begin{aligned}\sin 3A &= \sin(2A + A)\\ &= \sin 2A \cos A + \cos 2A \sin A\\ &= (2 \sin A \cos A) \cos A + (1 - 2\sin^2 A) \sin A\\ &= 2 \sin A (1 - \sin^2 A) + \sin A - 2\sin^3 A\\ &= 3 \sin A - 4 \sin^3 A\end{aligned}$$

Similarly,
$$\begin{aligned}\cos 3A &= \cos(2A + A)\\ &= \cos 2A \cos A - \sin 2A \sin A\\ &= (2\cos^2 A - 1) \cos A - (2 \sin A \cos A) \sin A\\ &= 2\cos^3 A - \cos A - (2 \sin^2 A \cos A)\\ &= 4\cos^3 A - 3 \cos A\end{aligned}$$

By using the ratios for known angles such as 15°, 30°, 45°, tests of the formulae for the double angle can be obtained.

Ex. 1. Given $\sin 30° = \frac{1}{2}$; find $\sin 60°$, $\tan 60°$.

$$\begin{aligned}\sin 60° &= 2 \sin 30° \cos 30°\\ &= 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.\end{aligned}$$

$$\tan 60° = \frac{2 \times \tan 30°}{1 - \tan^2 30°} = \frac{\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}} = \sqrt{3}.$$

Ex. 2. Given $\sin A = \frac{3}{5}$; find $\sin 2A$, $\cos 2A$, and $\tan 2A$.

$$\cos A = \pm\sqrt{1-\left(\frac{3}{5}\right)^2} = \pm\frac{4}{5}.$$

Taking the positive sign, then,

$$\sin 2A = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}.$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$= 1 - 2\times\frac{9}{25} = \frac{7}{25}.$$

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{24}{7}$$

The preceding formulae for multiple angles may be used to verify various trigonometrical identities

Ex. 3 Prove the following statements

(i) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A+B)$.

(ii) $\dfrac{\sin\theta + \sin(\theta+\phi) + \sin(\theta+2\phi)}{\cos\theta + \cos(\theta+\phi) + \cos(\theta+2\phi)} = \tan(\theta+\phi)$.

(i) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \dfrac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}$

$$= \tan\tfrac{1}{2}(A+B)$$

(ii) The given expression may be written

$$\frac{\{\sin(\theta+2\phi)+\sin\theta\}+\sin(\theta+\phi)}{\{\cos(\theta+2\phi)+\cos\theta\}+\cos(\theta+\phi)}$$

$$= \frac{2\sin(\theta+\phi)\cos\phi + \sin(\theta+\phi)}{2\cos(\theta+\phi)\cos\phi + \cos(\theta+\phi)}$$

$$= \frac{\sin(\theta+\phi)(1+2\cos\phi)}{\cos(\theta+\phi)(1+2\cos\phi)} = \tan(\theta+\phi).$$

It will be noticed that the sum or difference of any two sines, or cosines, can be obtained in the form of a product.

Ex. 4. $\sin 6A + \sin 4A = 2\sin\left(\dfrac{6A+4A}{2}\right)\cos\left(\dfrac{6A-4A}{2}\right)$

$$= 2\sin 5A\cos A.$$

Ex. 5. $\sin 5A - \sin 3A = 2\cos\left(\frac{5A+3A}{2}\right)\sin\left(\frac{5A-3A}{2}\right)$

$$= 2\cos 4A \sin A.$$

Similarly, $\cos 6A + \cos 4A = 2\cos 5A \cos A$,

and $\cos 3A - \cos 5A = 2\sin 4A \sin A$.

The preceding direct process must be clearly understood, then the converse process (*e.g.* given a product to obtain a sum or difference) will not present much difficulty.

Ex. 6. Express $2\sin 5A \cos A$ as the sum of two sines

Let $\sin x + \sin y = 2\sin 5A \cos A$.

Then $\frac{x+y}{2} = 5A$,

or $x + y = 10A$;

also $x - y = 2A$.

$$x = 6A,$$
$$y = 4A.$$

Hense, we obtain

$$\sin 6A + \sin 4A = 2\sin 5A \cos A.$$

Ex. 7 To show that $a = b\cos C + c\cos B$.

Given $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ say, and $A + B + C = 180^\circ$

Hence,

$$a = k\sin A, \quad \ldots \quad (1)$$
$$b = k\sin B, \quad \ldots \quad (2)$$
$$c = k\sin C. \quad \ldots \quad (3)$$

Multiplying (2) by $\cos C$ and (3) by $\cos B$ we have

$$b\cos C = k\sin B\cos C$$
$$c\cos B = k\sin C\cos B$$

adding

$$b\cos C + c\cos B = k(\sin B\cos C + \cos B\sin C)$$
$$= k\sin(B+C) = k\sin A,$$

because $\sin(B+C) = \sin A$;

$$b\cos C + c\cos B = k\sin A = a.$$

In like manner we can obtain

$$a\cos C + c\cos A = b$$
$$a\cos B + b\cos A = c.$$

EXERCISES. III.

1. Given $\cos A = \frac{3}{5}$, $\cos B = \frac{12}{13}$. Find $\sin(A+B)$ and $\cos(A+B)$.

2 The cosines of two angles of a triangle are $\frac{3}{5}$ and $\frac{12}{13}$ respectively; find the sine and cosine of the remaining angle

3. Prove that $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$.

4. Prove that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ + \cos 90^\circ = 0$.

5. From the relations $a = b\cos C + c\cos B$, $b = a\cos C + c\cos A$, $c = a\cos B + b\cos A$, show that $a^2 = b^2 + c^2 - 2bc\cos A$.

6. Write down the formulae for sine and cosine of the sum and difference of any two angles, and prove any one of them.

If $x = \sin^{-1} 0{\cdot}4848$ and $y = \tan^{-1} 0{\cdot}364$, find the value of $\cos(x+y)$.

7 If $\cos\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{4}{5}$, find the values of $\cos\dfrac{\alpha-\beta}{2}$ and $\cos^2\dfrac{\alpha+\beta}{2}$ the angles α and β being positive acute angles.

8. Prove the formula

$$\sin(A-B) = \sin A\cos B - \cos A\sin B,$$

and write down the corresponding formula for $\cos(A-B)$.

If $\sin A = 0{\cdot}8$ and $\sin B = 0{\cdot}6$, find the numerical values of $\sin(A-B)$ and $\cos(A-B)$.

9. Prove the formulae

(i) $\dfrac{\sin 3A - \sin A}{\cos 3A + \cos A} = \tan A$.

(ii) $4(\cos^3 10^\circ + \sin^3 20^\circ) = 3(\cos 10^\circ + \sin 20^\circ)$

10. A and B are the angles of a triangle. Given $\cos A = \frac{3}{4}$, show how to construct the angle A, and find the sine, tangent, and cotangent of A.

11. Show that

(i) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

(ii) $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$.

(iii) $\sin 70^\circ = \sin 10^\circ + \sin 50^\circ$.

12. If $\sin(A+B) = 0{\cdot}8$, and $\sin(A-B) = 0{\cdot}6$, find the value of $\tan 2A$.

13. Prove that

(i) $\sin 80^\circ = \sin 40^\circ + \sin 20^\circ$.

(ii) $\dfrac{\cos 2a + \cos 12a}{\cos 6a + \cos 8a} + \dfrac{\cos 7a - \cos 3a}{\cos a - \cos 3a} + 2\dfrac{\sin 4a}{\sin 2a} = 0$.

(iii) $\dfrac{\sin\alpha+\sin\beta+\sin(\alpha+\beta)}{\sin\alpha+\sin\beta-\sin(\alpha+\beta)}=\cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$

(iv) $\tan 4\theta=\dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}.$

14 Prove that

$$\frac{\tan^3\theta}{1+\tan^2\theta}+\frac{\cot^3\theta}{1+\cot^2\theta}=\frac{1-2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$$

Show that

15 $\cos\beta\cos(2\alpha+\beta)=\cos^2(\alpha+\beta)-\sin^2\alpha.$

16. $\dfrac{\cos x}{1-\tan x}+\dfrac{\sin x}{1-\cot x}=\sin x+\cos x.$

17. $2+4\cot^2 2A=\tan^2 A+\cot^2 A.$

18. $\tan(A+B)=\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}.$

19. (*a*) Find the numerical values of the sine and cosine of angles $22\frac{1}{2}^\circ$ and 75° respectively, (*b*) given $\sqrt{2}=1{\cdot}414$ and $\sqrt{6}=2{\cdot}449$, calculate the numerical value of $27+32\sin 195^\circ$.

20. Show that in a triangle ABC, $c=a\cos B+b\cos A$, when the angles A and B are acute, and when one of them (A) is obtuse. Given $a=6$, $b=5$, $c=10$, find $\cos C$, and from it find C.

21. Show that $\sin(A+B)=\sin A\cos B+\sin B\cos A$, using the relation $c=a\cos B+b\cos A$, having given $A+B+C=180^\circ$.

22. In the triangle ABC, if M is the middle point of BC, show that $4AM^2=b^2+c^2+2bc\cos A$

If BC is 6 inches long, find the length of AM, when

$$\tan C=5\tan B=9\cot A.$$

23. Show how the formula for $\tan(A+B)$ in terms of $\tan A$, $\tan B$, may be deduced from the formula for $\sin(A+B)$

24 Prove that $\cos(135^\circ+A)+\sin(135^\circ-A)=0.$

If $\tan A=\dfrac{\sqrt{3}}{4-\sqrt{3}}$, and $\tan B=\dfrac{\sqrt{3}}{4+\sqrt{3}}$, prove that $\tan(A-B)=0{\cdot}375$.

25. Assuming that

$$\left.\begin{aligned}\sin(A+B)&=\sin A\cos B+\cos A\sin B\\ \cos(A+B)&=\cos A\cos B-\sin A\sin B\end{aligned}\right\},$$

find in terms of the ratios of A the values of $\sin 2A$, $\cos 2A$, $\tan 2A$, $\sin\dfrac{A}{2}$, $\cos\dfrac{A}{2}$, and $\tan\dfrac{A}{2}$.

26. If $\cos\theta=\dfrac{3}{5}$, determine the values of $\cos 2\theta$, $\sin 2\theta$, $\cos\dfrac{\theta}{2}$.

CHAPTER IV.

TRIGONOMETRICAL EQUATIONS

Solution of Trigonometrical equations.—An equality of two expressions involving trigonometrical ratios, which is only true for certain definite values of an unknown angle, is called a **trigonometrical equation** The process of solving such an equation is in many respects similar to that adopted in an algebraical equation. The object is to find a value, or values, of the unknown angles which will satisfy the given equation

Having obtained such an equation in its simplest form, so that a trigonometrical ratio (such as sine, cosine, or tangent) is on the left of the equation and its numerical value on the right, the angle can be ascertained from Tables IV, V., VI. The process may be seen from the following examples

Ex. 1 What are the values of A less than 360° which satisfy the equation $2\cos A+1=0$

$$\text{Here } 2\cos A=-1;$$
$$\therefore \cos A=-\tfrac{1}{2},$$
$$\text{or } A=120^\circ \text{ or } 240^\circ.$$

The general value is given by

$$\theta \text{ rads } =(2n+1)\pi\pm\frac{\pi}{3}$$
$$\text{or } A^\circ=(2n+1)\,180^\circ\pm60^\circ.$$

Ex. 2. Find a series of values of A which satisfy the equation $\sin A=\frac{1}{3}$.

$$\sin A=\tfrac{1}{3}=0{\cdot}3333.$$

From Table IV. 0 3333 = sin 19° 28′.

Hence one angle is 19° 28′.

All the angles whose sine is $\frac{1}{3}$ may be obtained from the formula $2n\pi+(-1)^n\theta$.

Thus, when $n=0$ $A=19° 28'$,
,, $n=1$, $A=160° 32'$,
,, $n=2$, $A=379° 28'$,
,, $n=3$, $A=520° 28'$.
etc. etc.

Ex. 3. Solve the equation $\sin\theta+\cos\theta=\sqrt{2}$.

Dividing both sides of the equation by $\sqrt{2}$

$$\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=1\ ;$$

$$\therefore\ \sin\theta\cos 45°+\cos\theta\sin 45°=1\ ;$$

$$\therefore\ \sin\left(\theta+\frac{\pi}{4}\right)=1\ ;$$

$$\theta+\frac{\pi}{4}=\frac{\pi}{2}.$$

General value is $\qquad \theta+\frac{\pi}{4}=(4n+1)\frac{\pi}{2}.$

Hence $\qquad \theta=\frac{\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4},$ etc.

Ex 4. Solve the equation $\operatorname{cosec}\theta+\cot\theta=\sqrt{3}$;

$$\therefore\ \frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}=\sqrt{3},$$

$$\text{or}\quad \frac{1+\cos\theta}{\sin\theta}=\sqrt{3}$$

Now $\quad 1+\cos\theta=2\cos^2\frac{\theta}{2}$, and $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

Substituting

$$\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\sqrt{3}\ ;$$

$$\therefore\ \cot\frac{\theta}{2}=\sqrt{3},$$

$$\text{or}\quad \frac{\theta}{2}=\frac{\pi}{6}\ \text{or}\ \theta=\frac{\pi}{3},$$

and the general value obtained from $\frac{\theta}{2}=\frac{\pi}{6}+n\pi$.

$$\therefore\ \theta=2n\pi+\frac{\pi}{3}.$$

Ex. 5. Find all the positive values of θ not exceeding 180° which satisfy the following equations:

$$(a)\ 8\sin^3\theta - 7\sin\theta + \sqrt{3}\cos\theta = 0;$$
$$(b)\ \sin 3\theta + \cos 5\theta = \cos\theta.$$

(a) $8\sin^3\theta - 7\sin\theta + \sqrt{3}\cos\theta = 0.$ . . . . (i)

As $\qquad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (see p 32),

$$8\sin^3\theta = 6\sin\theta - 2\sin 3\theta.$$

Substituting in (i),

$$(6\sin\theta - 2\sin 3\theta) - 7\sin\theta + \sqrt{3}\cos\theta = 0;$$
$$\sqrt{3}\cos\theta - \sin\theta = 2\sin 3\theta \quad . . . . \ (ii)$$

Also $\qquad \sqrt{3}\cos\theta - \sin\theta = 2\sin(60° - \theta).$

Hence (ii) becomes

$$2\sin(60° - \theta) = 2\sin 3\theta;$$
$$3\theta = 60° - \theta + (-1)^n(2n+1)\pi,$$
$$\text{or} \quad 4\theta = 60° \quad \text{and} \quad \theta = 15°.$$

Other values are $\quad 3\theta = 180° - (60° - \theta); \qquad \theta = 60°;$

and $\qquad 3\theta = 360° + 60° - \theta; \qquad \theta = 105°.$

Hence the values are 15°, 60°, 105°

(b) $\sin 3\theta + \cos 5\theta = \cos\theta$,

$$\sin 3\theta = \cos\theta - \cos 5\theta$$
$$= 2\sin 3\theta \sin 2\theta. \quad . . . . \ \text{(i) (p 28)}$$

The value $\sin 3\theta = 0$ will satisfy this equation;

$$\therefore 3\theta = 0, 180° \text{ or } 360°,$$
$$\text{or} \quad \theta = 0, \quad 60°, \quad 120°.$$

Dividing by $\sin 3\theta$, we obtain

$$1 = 2\sin 2\theta; \qquad \sin 2\theta = \tfrac{1}{2},$$
$$\text{or} \quad 2\theta = 30° \text{ or } 150°,$$
$$\theta = 15° \text{ or } 75°$$

Hence, the values are 0°, 15°, 60°, 75°, 120°.

Elimination.—In trigonometrical, as in algebraical equations, from a sufficient number of distinct and independent equations one or more unknown terms may be eliminated. For this purpose the relations between trigonometrical ratios, such as $\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta = 1 + \tan^2\theta$, etc., are very important. The following examples will serve to illustrate some of the processes which may be adopted

Ex. 1. Eliminate θ between the equations

$$a \sin\theta + b\cos\theta = m; \qquad \ldots\ldots(i)$$
$$a\cos\theta - b\sin\theta = n \qquad \ldots\ldots(ii)$$

Square equation (i), then

$$a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = m^2. \qquad \ldots\ldots(iii)$$

Similarly squaring (ii),

$$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = n^2 \qquad \ldots\ldots(iv)$$

Adding (iii) and (iv),

$$a^2(\sin^2\theta + \cos^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = m^2 + n^2;$$
$$\therefore\ a^2 + b^2 = m^2 + n^2$$

Ex 2 Eliminate ϕ between the equations

$$x = 2b\cos\phi\cos 2\phi \quad b\cos\phi; \qquad (i)$$
$$y = 2b\cos\phi\sin 2\phi - b\sin\phi \qquad (ii)$$

Divide each equation by b, then

$$\frac{x}{b} = 2\cos\phi\cos 2\phi - \cos\phi = (\cos 3\phi + \cos\phi) - \cos\phi = \cos 3\phi,$$

$$\frac{y}{b} = 2\cos\phi\sin 2\phi - \sin\phi = (\sin 3\phi + \sin\phi) - \sin\phi = \sin 3\phi.$$

Square and add,

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} = \cos^2 3\phi + \sin^2 3\phi = 1;$$
$$\therefore\ x^2 + y^2 = b^2.$$

Ex. 3. Given $p^2 + q^2 = \sin^2\theta$

Show that $p^2 + \left(\dfrac{pq}{1+\cos\theta}\right)^2 + \dfrac{(q^2 + \cos\theta + \cos^2\theta)^2}{(1+\cos\theta)^2} = 1;$

$$i.e.\ (1+\cos\theta)^2(p^2-1) + p^2q^2 + (q^2 + \cos\theta + \cos^2\theta)^2 = 0.$$
$$p^2 + q^2 = 1 - \cos^2\theta;$$
$$\therefore\ q^2 + \cos^2\theta = 1 - p^2;$$

hence $\quad (1+\cos\theta)^2(p^2-1) + p^2q^2 + (q^2 + \cos\theta + \cos^2\theta)^2$

$$= (1+\cos\theta)^2(p^2-1) + p^2(1 - p^2 - \cos^2\theta) + (1 - p^2 + \cos\theta)^2$$
$$= (1+\cos\theta)^2(p^2-1) + p^2(1-\cos^2\theta) - p^4 + (1+\cos\theta)^2 - 2p^2(1+\cos\theta) + p^4$$
$$= (1+\cos\theta)^2p^2 + p^2(1-\cos^2\theta) - 2p^2(1+\cos\theta)$$
$$= p^2\{1 + 2\cos\theta + \cos^2\theta + 1 - \cos^2\theta - 2 - 2\cos\theta\} = 0.$$

Ex. 4. If

$$\left.\begin{array}{l} p = 1 + \sin^2\theta \\ q = 1 + \cos^2\theta \end{array}\right\}, \qquad \ldots(i)$$

Show that $2(p^3 + q^3) + 9q^2 = 27(1 + \cos^4\theta)$.

From (1) we obtain $p + q = 3$.

Also $\quad p = 1 + 1 - \cos^2\theta = 2 - \cos^2\theta,$

$$q = 1 + \cos^2\theta.$$

Multiplying $\qquad pq = 2 + \cos^2\theta - \cos^4\theta$;

$$\therefore\ 2(p^3+q^3)+9q^2 = 2(p+q)(p^2-pq+q^2)+9q^2$$
$$= 6(4 - 4\cos^2\theta + \cos^4\theta - 2 - \cos^2\theta + \cos^4\theta + 1 + 2\cos^2\theta + \cos^4\theta)$$
$$+ 9(1 + 2\cos^2\theta + \cos^4\theta)$$
$$= 6(3 - 3\cos^2\theta + 3\cos^4\theta) + 9(1 + 2\cos^2\theta + \cos^4\theta)$$
$$= 27(1 + \cos^4\theta).$$

EXERCISES. IV.

Find values less than 180° which will satisfy each of the following equations:

1 $5\tan^2 x - \sec^2 x = 11$

2 $2\cos 4A \sin A = \sqrt{2}\cos 4A$

3. $\cos^2 A + 2\sin^2 A - \frac{5}{2}\sin A = 0$

4. $\tan A + 3\cot A = 4$.

5. $2\sin^2 A - 5\cos A = 4$

6. $\sin 7x - \sin x = \sin 3x$.

7 (i) $17\sin\theta = 15\sin 63° 18'$; (ii) $\cos\theta = \cos 37° 59' \cos 153° 18'$; (iii) $\tan 2\theta = -\sin 52° 2'$.

8. $2\sin^2 A - (1+\sqrt{3})\sin 2A + 2\sqrt{3}\cos^2 A = 0$.

9. $\sin^2 x + \cos^2 x = 3\cos x$.

10. $\cos x + \sqrt{3}\sin x = 1$.

11. $4\tan x = \sqrt{3}\sec^2 x$

12. $\tan x \tan 2x = 1$

13. $\tan^2 x - (1+\sqrt{3})\tan x + \sqrt{3} = 0$

14. $\cos 3A + \cos 5A + \sqrt{2}(\cos A + \sin A)\cos A = 0$

15. What is the value of θ less than 360° which satisfies the equations; $\qquad 5\sin\theta + 3 = 0$, *and* $5\cos\theta + 4 = 0$.

16 Find a value of θ which satisfies the equation

$$\sin\theta + 2\cos\pi + 4\tan\frac{\pi}{4} = 1.$$

17 If $\cos 41° 24' = \frac{3}{4}$ find an angle θ which satisfies the equation

$$4\cos 2\theta + 3 = 0.$$

18. Find the value or values of θ less than 180° which satisfy the equations·

(i) $2\cos\theta + 1 = 0$, $\quad$ (ii) $\tan\theta + 1 = 0$, $\quad$ (iii) $13\sin\theta = 3$.

19. Find the values of θ between 0° and 180° which satisfy the equation $\qquad \tan^4\theta - 4\tan^2\theta + 3 = 0$.

20. The sine of 26° 24′ = 0·4446. Write down the values of $\cos 243° 36'$ and $\sin 333° 36'$.

21. Find the four least positive values of θ which satisfy the equation $2\tan^2 2\theta = 4{\cdot}5$.

22. Calculate the values of θ between $0°$ and $360°$ which satisfy the equation $1{\cdot}7\tan^2\theta - 14{\cdot}4 = 0$.

23. It is known that A and B are each less than $90°$. If

$$A = \tan^{-1}\frac{5}{6} \text{ and } \tan 2B = \sqrt{2{\cdot}16\dot{5}}$$

find the values of A and B correct to the nearest minute.

24. Find the least positive value of B which satisfies the equation

$$24\tan^2 B - 15 = 0$$

25 Find a positive value of θ less than $180°$ which will satisfy the equation

$$\sin\theta = \frac{h}{2a}\left(\frac{w}{w - w'}\right)^{\frac{1}{2}}$$

$$\text{when } \frac{h}{a} = \frac{3121}{4183} \text{ and } \frac{w'}{w} = \frac{719}{1719}$$

26. Solve the equation

$$5\tan^2 x + \sec^2 x = 7.$$

27. Calculate the value of θ less than $180°$ which satisfies the equation $\cos\theta = \cos 45° \cos 139° \, 6'$

28 Find all the positive values of θ less than $360°$ which satisfy the equation $4\sin^2\theta - 2\sin\theta - 1 = 0$

29. Show that $8(\sin^2 42° - \cos^2 78°) = \sqrt{5} + 1$.

30. Find a value of θ which will satisfy each of the following equations. (*a*) $2\sin^2\theta = 3\cos\theta$, (*b*) $1 + 2\sin^2\theta = 2\cos^2\theta$

31. Determine the least value of ϕ which will satisfy the equation

$$\sqrt{3}\tan^2\phi + 1 = (1 + \sqrt{3})\tan\phi.$$

32. Find the four least positive values of A and B such that $\sin A = \frac{1}{2}$ and $24\tan^2 2B - 15 = 0$.

33 Prove that $\cos 9° - \sin 39° - \cos 69° + \sin 99° = \sin\frac{9\pi}{20}$

34. Find the least positive value of B which satisfies the equation

$$24\tan^2 2B - 14{\cdot}97 = 0.$$

35. If $4\cot 2\theta = \cot^2\theta - \tan^2\theta$, prove that all possible values of θ are given by

$$\theta = n\pi \pm \frac{\pi}{4}$$

36. Find a value of θ which satisfies the equation

$$5\cos\theta + 7\sin\theta = 5{\cdot}915.$$

37. Find the values of A which satisfy the equation

$$\cos 8A - \cos 5A + \cos 3A = 1.$$

CHAPTER V.

INDICES. LOGARITHMS.

Indices.—The letter or number, placed near the top and to the right of a quantity, which expresses the power of a quantity, is called the index. Thus, in a^5, a^7, a^9, the numbers 5, 7, and 9, are called the **indices** of a, and are read as "a to the power five," "a to the power seven," etc. Similarly a^b denotes a to the power b. There are three index rules or laws.

First index rule.—To multiply together different powers of the same quantity, add the index of one to the index of the other. To divide different powers of the same quantity, subtract the index of the divisor from the index of the dividend.

$$\text{Thus, } a^3 \times a^2 = (a \times a \times a)\,(a \times a) = a^{3+2} = a^5.$$

Ex. 1. $a^3 \times a^5 = a^{3+5} = a^8$

Ex. 2. $a^2 \times a^3 \times a^4 = a^{2+3+4} = a^9$.

These results may be expressed in a more general manner as follows:

$$a^m = (a \times a \times a \text{ ..to } m \text{ factors})$$

and $$a^n = (a \times a \times a\text{. .to } n \text{ factors}),$$

$$\therefore a^m \times a^n = (a \times a \times a \text{ ..to } m \text{ factors})\,(a \times a \times a \text{...to } n \text{ factors})$$
$$= (a \times a \times a\text{. .to } m+n \text{ factors})$$
$$= a^{m+n}.$$

This most important rule has been shown to be true when $m=3$ and $n=5$. Other values of m and n should be assumed, and a further verification obtained.

Also $$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a^{5-3} = a^2.$$

Similarly $$\frac{a^m}{a^n} = \frac{a \times a \times a \text{ to } m \text{ factors}}{a \times a \times a \text{ to } n \text{ factors}} = a^{m-n}.$$

In like manner, the product of any number of positive or negative integers m, n, p,... is given by

$$a^m \times a^n \times a^p \ldots = a^{m+n+p+\ldots}$$

It is often found convenient to use both fractional and negative indices in addition to those described.

The meaning attached to fractional and negative indices is such that the previous rule holds for them also When one fractional power of a quantity is multiplied by another fractional power, the fractional indices are added; and when one fractional power is divided by another the fractional index of the former is subtracted from that of the latter.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a,$$

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{2}{3}}\ ;\ a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$$

Hence, the meaning to attach to $a^{\frac{1}{2}}$ is the square root of a; to $a^{\frac{2}{3}}$ is the cube root of a squared; and to $a^{\frac{1}{3}}$ the cube root of a.

Thus, $\sqrt{a}$ can be written as $a^{\frac{1}{2}}$,

$\sqrt[3]{a}$ can be written as $a^{\frac{1}{3}}$.

Also, $$\frac{1}{\sqrt{a}} = a^{-\frac{1}{2}},$$

and $$\frac{1}{\sqrt[3]{a}} = a^{-\frac{1}{3}}$$

Again, $$\frac{a^{\frac{1}{3}}}{a^{\frac{1}{2}}} = a^{\frac{1}{3}} \times a^{-\frac{1}{2}} = a^{\frac{1}{3}-\frac{1}{2}} = a^{-\frac{1}{6}}.$$

Also, $$\frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}}} = a^{\frac{1}{3}-\frac{1}{3}} = a^0.$$

Similarly, $$\frac{a^3}{a^3} = \frac{a \times a \times a}{a \times a \times a} = a^{3-3} = a^0$$

Generally, since $a^m \times a^n = a^{m+n}$ is true for all values of m and n. If n be 0, then

$$a^m \times a^0 = a^{m+0} = a^m\ ;$$

$$\therefore\ a^0 = \frac{a^m}{a^m} = 1.$$

Again $\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdots$ to n factors $= \frac{a^n}{b^n}$.

If $a=1$, then $\left(\frac{1}{b}\right)^n = \frac{1}{b^n}$.

Similarly $a^m \times a^{-m} = \frac{a^m}{a^m} = a^0 = 1$.

Hence, any quantity raised to the power 0 is equal to 1.

Second index rule.—To obtain a power of a power, multiply the two indices.

Ex. 1. To obtain the cube of a^2 we have

$$(a^2)^3 = (a \times a)(a \times a)(a \times a) = a^{2\times 3} = a^6,$$

where the index is the product of the indices 2 and 3.

Ex. 2. Find the value of $(2{\cdot}15^2)^3$.

$$(2{\cdot}15^2)^3 = 2{\cdot}15^{2\times 3}$$
$$= 2{\cdot}15^6 = 98{\cdot}72,$$

or, expressing this rule as a formula,

$$(a^m)^n = a^{mn},$$

$\therefore$ *a quantity a^m may be raised to a power n by using as an index the product mn.*

To show that $(a^m)^n = a^{mn}$.

$$(a^m)^n = a^m \times a^m \ldots \text{ to } n \text{ factors};$$

but each a^m contains a repeated m times, therefore

$$(a^m)^n = a \times a \ldots \text{ to } mn \text{ factors},$$
$$(a^m)^n = a^{mn}.$$

If we assume m to be 4 and n to be 2,

$$(a^m)^n = (a^4)^2 = (a \times a \times a \times a)(a \times a \times a \times a)$$
$$= a^{4\times 2} = a^8.$$

Ex. 3. Which is greater $\sqrt{3}$ or $\sqrt[3]{5\frac{1}{5}}$?

Raise each of the given quantities to the sixth power;

$$\therefore\ (3^{\frac{1}{2}})^6 = 3^3 = 27$$
$$\left\{(5\tfrac{1}{5})^{\frac{1}{3}}\right\}^6 = (5\tfrac{1}{5})^2 = (\tfrac{26}{5})^2 = 27{\cdot}04.$$

Hence $\sqrt[3]{5\frac{1}{5}}$ is greater than $\sqrt{3}$.

Third index rule.—To raise a product to any power raise each factor to that power.

Ex. 1. $(abcd)^m = a^m \times b^m \times c^m \times d^m$.

Ex. 2. Let $a=1$, $b=2$, $c=3$, $d=4$, and $m=2$

Then $$(abcd)^m = (1 \times 2 \times 3 \times 4)^2 = 1^2 \times 2^2 \times 3^2 \times 4^2$$
$$= 24^2 = 576.$$

In fractional indices, the index may be written either in a fractional form or the root symbol may be used. The general form is $a^{\frac{m}{n}}$. This may be written in the form $\sqrt[n]{a^m}$, which is read as *the n^{th} root of a to the power m*

Ex. 3. $2^{\frac{5}{3}} = \sqrt[3]{2^5} = \sqrt[3]{32} = 3 \cdot 174$.

Ex 4. Find the values of $8^{\frac{2}{3}}$, $64^{-\frac{1}{2}}$, $4^{-\frac{3}{2}}$.

Here $$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4.$$
$$64^{-\frac{1}{2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$
$$4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

Ex. 5. Find the value of $64^{\frac{1}{2}} + 4^{1.5} + 2^{2.5} + 27^{\frac{1}{3}}$.

Here $$64^{\frac{1}{2}} = 8,\quad 4^{1.5} = 4^{\frac{3}{2}} = 64^{\frac{1}{2}} = 8,$$
$$2^{2.5} = 2^{\frac{5}{2}} = 32^{\frac{1}{2}} = 5 \cdot 656,$$
$$27^{\frac{1}{3}} = 3.$$

Hence $$64^{\frac{1}{2}} + 4^{1.5} + 2^{2.5} + 27^{\frac{1}{3}} = 24.656.$$

Ex. 6. Find, to two places of decimals, the value of $x^2 - 5x^{\frac{1}{2}} + x^{-2}$, when $x=5$.

Here $$x^2 - 5x^{\frac{1}{2}} + x^{-2} = 25 - 5\sqrt{5} + \frac{1}{5^2}$$
$$= 25 - \frac{10}{2} \times 2 \cdot 236 + 0 \cdot 04 = 13 \cdot 86.$$

Ex. 7. Solve the equations

$$\frac{27^x}{9^y} = 1 \quad \ldots\ldots \quad (i) \qquad \frac{81^y}{3^x} = 243 \quad \ldots \quad (ii)$$

From (i) $\therefore\ 3^{3x} = 3^{2y}$; $\therefore\ 3x = 2y. \ldots\ldots \ldots$ (iii)

From (ii) $3^{4y} = 3^x \times 3^5 = 3^{x+5}$;

$\therefore\ 4y = x + 5. \ldots\ldots\ldots\ldots$ (iv)

Combining (iii) and (iv) $3x = 12y - 15 = 2y$;

$$\therefore\ 10y = 15,\quad y = \frac{3}{2},\quad x = 1.$$

EXERCISES. V.

1. Simplify $\dfrac{\left(\frac{3}{2}\right)^{\frac{1}{2}} - \left(\frac{3}{2}\right)^{\frac{3}{2}}}{6^{\frac{1}{2}} + \left(\frac{2}{3}\right)^{\frac{1}{2}}}$.

2 Show that $x^{\frac{2}{3}} + y^{\frac{2}{3}} + 4z^2$ is one of the factors of

$$x^2 + y^2 - 4z^2\,(3x^{\frac{2}{3}}y^{\frac{2}{3}} - 16z^4).$$

3 Multiply together $x^{\frac{1}{n}} - x^{-\frac{1}{n}}$ and $x^{\frac{2}{n}} + 1 + x^{-\frac{2}{n}}$

4 Divide

$$x^{12} + \frac{1}{x^{12}} + 6\left(x^8 + \frac{1}{x^8}\right) + 15\left(x^4 + \frac{1}{x^4}\right) + 20 \text{ by } x^6 + \frac{1}{x^6} + 3\left(x^2 + \frac{1}{x^2}\right).$$

5. Express $\sqrt{x} + \sqrt[4]{(xy)} + \sqrt{y}$ with fractional indices and multiply it by $x^{-\frac{1}{2}} + x^{-\frac{1}{4}}y^{-\frac{1}{4}} + y^{-\frac{1}{2}}$.

Simplify

6 $\sqrt[6]{(a^3b\sqrt[5]{a^3bc})^5}$

7. $[a^{-1}b\{a^{-4}b^3(a^3b\sqrt{ab})^2\}^{\frac{1}{3}}]^{-1}$.

8. Solve the equations

$$18y^x - y^{2x} = 81\,;$$
$$3^x = y^2.$$

9. (*a*) Assuming that $a^m \times a^n = a^{m+n}$ is true for all values of m and n, find the meaning of the symbols a^{-4} and $a^{-\frac{1}{4}}$

(*b*) Simplify $\qquad (x^{-\frac{1}{3}}y^{-\frac{1}{4}})^{-6}$

(*c*) Find the product of

$$x^{-\frac{1}{6}}y^{-\frac{2}{3}} \text{ and } \frac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{\sqrt[6]{x^2y^3}} \div x^{\frac{1}{6}}y^{-\frac{1}{6}}.$$

10 Divide $x - 256y^3$ by $4x^{-\frac{1}{4}} + y^{-\frac{3}{4}}$.

11. Multiply $a + b^{\frac{2}{3}} + c^{\frac{1}{2}} - b^{\frac{1}{3}}c^{\frac{1}{4}} - c^{\frac{1}{4}}a^{\frac{1}{2}} - a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{3}} + c^{\frac{1}{4}}$.

12. (i) Prove that

$$\frac{x^{\frac{3}{2}} + y^{\frac{3}{2}} + xy(x^{-\frac{1}{2}} + y^{-\frac{1}{2}})}{x^{\frac{3}{2}} - y^{\frac{3}{2}} - xy(x^{-\frac{1}{2}} - y^{-\frac{1}{2}})} = \frac{x+y}{x-y}$$

(ii) Find the value of

$$x^3 + 2y^3 + 2z^3 + 6xyz, \text{ when } x = y + z = \sqrt[3]{4}.$$

13 Find the value of $1 + 2^{-2} + 2^{-3} \times 5^{-1} + 2^{-7} + 2^{-8}$.

Simplify the following expressions:

14. $\left(\dfrac{a^{\frac{3}{4}}b^{\frac{7}{4}}}{a^{\frac{2}{3}}b^{\frac{1}{2}}}\right)^{-\frac{1}{5}} \times \left\{\sqrt[3]{(a^{-2})}\,\sqrt[6]{(b^{-1})}\right\}^2.$ **15** $\left\{ab^2(ab^3)^{\frac{1}{2}}(a^2b^3)^{\frac{1}{6}}\right\}^{\frac{1}{2}}.$

16 (i) $\dfrac{pq^{-1}+p^{-1}q+2}{p^{\frac{1}{3}}q^{-\frac{1}{3}}+p^{-\frac{1}{3}}q^{\frac{1}{3}}-1}.$ (ii) $\sqrt{(x^{-\frac{5}{3}}y^{3}z^{-\frac{2}{3}})} \div \sqrt[3]{(x^{\frac{1}{2}}y^{4}z^{-1})}$

17. $(x^{-\frac{1}{3}}y^{-\frac{1}{4}})^{-6} \times y^{-\frac{3}{2}}.$

18. $(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{3}{4}} + b^{\frac{3}{2}})(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{3}{4}} + b^{\frac{3}{2}})$, and find its value when $a=3$, $b=4$.

19. Find the value of $\sqrt{\dfrac{5}{x}} - \sqrt[3]{-x}$, when $x=0{\cdot}008$.

Logarithms.—Logarithms of numbers consist of an integral part which may be positive, negative, or zero, called the index or characteristic, and a decimal part called the mantissa. Referring to Table II. the reader will find that opposite each of the numbers from 10 to 99 four figures are placed; these are *positive numbers* and each set of four is called a *mantissa*.

The *characteristic* has to be supplied when writing down the logarithm of any given number. Logarithmic tables have been calculated for all numbers from 1 to 100,000 giving seven or more figures in the mantissa, but for all practical purposes the numbers in such a table as that referred to, and known as *four-figure logarithms*, are very convenient.

By means of the numbers 10 to 99 in the left hand column with (*a*) those along the top of the table, and (*b*) those in the difference column on the right, the logarithm of any number consisting of four significant figures can be written down.*

* The numerical values of logarithms increase much more rapidly and the numbers in the difference columns are greater in the earlier part of Table II. than elsewhere, and there is more liability to error here than at any other place. Several methods may be devised to make such a table uniformly accurate, one is to calculate two or more columns of differences for each of the ten horizontal rows (10-20). Another method is as follows: Let N denote a given number, write down $\log \dfrac{N}{2}$, and finally add $\log 2$.

Ex. Find $\log 11{\cdot}78$.

Using seven figure logarithms, $\log 11{\cdot}78 = 1{\cdot}0711453$

From Table II., $\log 11{\cdot}78 = 1{\cdot}0712$, the last figure is in error.

Using the rule: $\dfrac{11{\cdot}78}{2} = 5{\cdot}89$; $\therefore \log 5{\cdot}89 + \log 2 = 1{\cdot}0711$.

In logarithms, all numbers are expressed as the powers of some number called the **base**.

The logarithm of a number to a given base is the index showing the power to which that base must be raised to give the number.

Let N denote any number, and a the given base, then if by raising a to some power x we can obtain N,

$$N = a^x \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Thus, if the base be 2, then $2^3 = 8$; or, 3 is the logarithm of 8 to the base 2. This can be expressed as $\log_2 8 = 3$.

Also, as $64 = 2^6 = 4^3 = 8^2$.

Hence 6 is the log of 64 to the base 2,
3 " " " 4,
2 " " " 8.

These facts may, as just indicated, be expressed thus:

$$\log_2 64 = 6, \quad \log_4 64 = 3, \quad \log_8 64 = 2,$$

using in each case the abbreviation log for logarithm.

Characteristic and Mantissa.—As will be seen from the preceding paragraphs any number can be used as base; but the system of logarithms in which the base is 10 (known as common logarithms) is that generally used. It is then only necessary to print in a table the decimal part, or mantissa; the characteristic can be written by inspection.

As the base is 10, Eq. (1) above may be written

$$N = 10^x;$$

$$\therefore \log_{10} N = x.$$

Substituting powers of 10 for N,

$$1 = 10^0; \qquad \log 1 = 0.$$

Also $10 = 10^1; \quad \therefore \log 10 = 1.$

Again $100 = 10^2; \quad \log 100 = 2.$

Again as 0·1, 0·01, and 0·001 can be written in the form $\frac{1}{10}$ or 10^{-1}, $\frac{1}{100}$ or 10^{-2}, $\frac{1}{1000}$ or 10^{-3} respectively,

$$\therefore \log 0{\cdot}1 = \log 10^{-1} = -1,$$

$$\log 0{\cdot}01 = \log 10^{-2} = -2,$$

and $\log 0{\cdot}001 = \log 10^{-3} = -3.$

The **mantissa** in the tables is always a positive number. In order, therefore, to preserve its character, and to indicate that the negative sign attaches to the characteristic alone, we write the negative sign over the characteristic. Thus, $\log 0{\cdot}1$ is not written -1 but as $\bar{1}$, and $\log 0{\cdot}01 = \bar{2}$. In the preceding cases only the characteristic has been inserted, for each mantissa consists of a series of ciphers.

$$\log 1 = 0{\cdot}0000$$
$$\log 10 = 1{\cdot}0000$$
$$\log 100 = 2{\cdot}0000$$
$$\log 0{\cdot}01 = \bar{2}{\cdot}0000, \text{ etc.}$$

As the logarithm of 1 is 0, and log 10 is 1, it is clear that the logarithms of all numbers between 1 and 10 will consist only of a series of figures after the decimal point. Thus, $\log 3 = 0{\cdot}4771$ indicates that if we raise 10 to the power $0{\cdot}4771$ we obtain 3, or $10^{0{\cdot}4771} = 3$.

In a similar manner, the logarithm of 300 might be written as $10^2 \times 10^{0{\cdot}4771}$,

$$\log 300 = 10^{2{\cdot}4771}.$$

Thus, we write $\qquad \log 300 = 2{\cdot}4771$

Similarly, $\qquad 0{\cdot}0003 = \frac{3}{10000} = 3 \times 10^{-4}$,

$$0{\cdot}0003 = 10^{\bar{4}{\cdot}4771},$$

or $\qquad \log 0{\cdot}0003 = \bar{4}{\cdot}4771.$

The most convenient rule by which the characteristic may be found is as follows. **The characteristic of any number greater than unity is positive, and is less by one than the number of figures to the left of the decimal point. The characteristic of a number less than unity is negative, and is greater by one than the number of zeros which follow the decimal point.**

Ex. Write down log 30 and $\log 0{\cdot}00003$

Here $\qquad \log 30 = 1{\cdot}4771,$

and $\qquad \log 0{\cdot}00003 = \bar{5}{\cdot}4771$

Multiplication.—Add the logarithms of the multiplier and multiplicand together; the sum is the logarithm of their product. The number corresponding to this logarithm, called the antilog, is the product required.

Let a and b denote two numbers.

Let $\log a = x$ and $\log b = y$;

$$\therefore\ a = 10^x,\ b = 10^y,$$

$$a \times b = 10^{x+y},$$

$$\text{or}\quad \log_{10} ab = x + y = \log a + \log b.$$

Ex. 1. Multiply $0{\cdot}03056 \times 0{\cdot}4105$.

$$
\begin{array}{lr}
\text{From Table II}, \quad \log 305 = & 4843 \\
\text{Diff. col. for } 6 & 9 \\
\hline
\log 0{\cdot}03056 = & \bar{2}{\cdot}4852
\end{array}
$$

$$
\begin{array}{lr}
\text{Similarly,} \quad \log 0{\cdot}4105 = & \bar{1}{\cdot}6133 \\
\hline
\text{log of product} = & \bar{2}{\cdot}0985
\end{array}
$$

$$
\begin{array}{lr}
\text{From Table III.,} \quad \text{antilog} & 1253 \\
\text{Diff. col. for } 5, & 1 \\
\hline
\therefore\ \text{antilog } 0985 = & 1254
\end{array}
$$

The numerical part of the product is 1254, and the characteristic is $\bar{2}$.

Hence $0{\cdot}03056 \times 0{\cdot}4105 = 0{\cdot}01254$

Division.—Subtract the logarithm of the divisor from the logarithm of the dividend and the result is the logarithm of the quotient of the two numbers. The number corresponding to this logarithm is the quotient required

Let a and b be the two numbers.

Let $\log a = x$ and $\log b = y$;

$$a = 10^x \qquad b = 10^y.$$

Hence
$$\frac{a}{b} = \frac{10^x}{10^y} = 10^{x-y},$$

$$\text{or } \log \frac{a}{b} = x - y = \log a - \log b.$$

Ex. 1. Divide $30{\cdot}56$ by $4{\cdot}105$.

Let z denote the value required;

$$\log z = \log 30{\cdot}56 - \log 4{\cdot}105$$

$$= 1{\cdot}4852 - 0{\cdot}6133 = 0{\cdot}8719;$$

$$\therefore\ z = 7{\cdot}446.$$

Hence $30{\cdot}56 \div 4{\cdot}105 = 7{\cdot}446$.

Involution.—**To obtain the power of a number, multiply the logarithm of the number by the index representing the power required; the product is the logarithm of the number required.**

Let $\log a = x.$

Then $a = 10^x.$

And $a^n = (10^x)^n = 10^{xn};$

$$\therefore \log_{10} a^n = nx = n \log a.$$

Ex. 1. Find the value of $4{\cdot}105^{1{\cdot}23}$.

Let z denote the value required.

$$\log z = 1{\cdot}23 \log 4{\cdot}105 = 1{\cdot}23 \times 0{\cdot}6133$$
$$= 0{\cdot}7544 = \log 5{\cdot}680;$$
$$z = 5{\cdot}680.$$

It should be carefully noticed that the logarithm of a decimal number consists of a negative characteristic and a positive mantissa.

Evolution.—**To obtain the root of a number, divide the logarithm of the number by the number which indicates the root.**

Ex. 1. Find the cube root of 32·4.

Let x denote the value;

$$\therefore x = (32{\cdot}4)^{\frac{1}{3}};$$
$$\log x = \tfrac{1}{3} \log 32{\cdot}4 = \tfrac{1}{3} \times 1{\cdot}5105 = 0{\cdot}5035 = \log 3{\cdot}188;$$
$$\therefore x = 3{\cdot}188.$$

No difficulty will be experienced when, as in the preceding example, the characteristic and mantissa are both positive. But, as already indicated, although the characteristic may be negative, the mantissa remains positive, and a little alteration in form is necessary, in order to make such a logarithm exactly divisible by the number.

Ex. 2. Find the fifth root of 0·0324.

Assume $x = (0{\cdot}0324)^{\frac{1}{5}};$

$$\log 0{\cdot}0324 = \bar{2}{\cdot}5105.$$

To make this exactly divisible we increase the characteristic to $\bar{5}$, and make the necessary correction. Thus,

$$\bar{2}{\cdot}5105 = \bar{5} + 3{\cdot}5105.$$

Hence $\log x = \tfrac{1}{5}(\bar{5} + 3{\cdot}5105) = \bar{1}{\cdot}7021 = \log 0{\cdot}5036;$

$$\therefore x = 0{\cdot}5036$$

The alteration may be made as suggested; but, after a little practice, the steps indicated are most easily carried out mentally. To extract say the $1\cdot065^{\text{th}}$ root of $0\cdot0324$, it is advisable to make the mantissa of the logarithm negative in order to carry out the division indicated and finally to make the mantissa positive before referring to the table of antilogs for the result.

When it is required to raise a number less than unity to a negative power, it will usually be found most convenient to make the mantissa of the logarithm negative before proceeding to multiply.

Ex. 3. Calculate the value of $0\cdot04105^{-2\cdot3}$.

$\text{Log}\,0\cdot04105 = \bar{2}\cdot6133$, in which the characteristic is negative, but the mantissa is positive. When both are made negative

$$\bar{2}\cdot6133 = -2 + 0\cdot6133 = -1\cdot3867;$$

Let x denote the value required.

$$\log x = -2\cdot3 \times (-1\cdot3867) = 3\cdot1894 = \log 1546;$$
$$x = 1546.$$

Ex. 4. Compute the value of $(5)^a + (3)^b + (0\cdot042)^c$, where $a = 2\cdot43$, $b = -0\cdot246$ and $c = 0\cdot476$.

Let x denote the value required. Then substitute the given values,

$$x = 5^{2\cdot43} + 3^{-0\cdot246} + 0\cdot042^{0\cdot476}.$$

As the three terms are connected by the signs of addition it is necessary to evaluate each separately and afterwards to add.

Thus, $\log 5^{2\cdot43} = 2\cdot43 \log 5 = 0\cdot6990 \times 2\cdot43 = 1\cdot6986 = \log 49\cdot96;$

$$\therefore\ 5^{2\cdot43} = 49\cdot96$$

$$\log 3^{-0\cdot246} = -0\cdot246 \log 3 = 0\cdot4771 \times (-0\cdot246)$$
$$= -0\cdot1174 = \bar{1}\cdot8826 = \log 0\cdot7632;$$
$$3^{0\cdot246} = 0\cdot7632.$$

Again, $\log 0\cdot042 = \bar{2}\cdot6232 = -1\cdot3768.$

Hence, $\log 0\cdot042^{0\cdot476} = -1\cdot3768 \times 0\cdot476 = -0\cdot6554$

$$= \bar{1}\cdot3446 = \log 0\cdot2211;$$
$$\therefore\ 0\cdot042^{0\cdot476} = 0\cdot2211.$$

Adding all the separate terms

$$x = 49\cdot96 + 0\cdot7632 + 0\cdot2211 = 50\cdot94.$$

Napierian logarithms.—The system of logarithms employed by Napier, the discoverer of logarithms, and called the **Napierian** or **Hyperbolic system**, is used in all theoretical investigations and very largely in practical calculations The base of this system is the number which is the sum of the series

$$1+1+\tfrac{1}{2}+\frac{1}{2\times 3}+\frac{1}{2\times 3\times 4}+\dots \text{ (p. 289)},$$

this sum to five figures is 2·7183 Usually the letter e is used to denote this number, as for example log 2 to base 10 would be written $\log_{10}2$ or more simply as $\log 2$, but the hyperbolic logarithm of 2 is written as $\log_e 2$

Transformation of logarithms.—A system of logarithms calculated to a base a may be transformed into another system in which the base is b

Let N be a number. Its logarithms in the first system we may denote by x and in the second system by y.

$$\text{Then} \qquad N=a^x=b^y \text{ or } b=a^{\frac{x}{y}};$$

$$\frac{x}{y}=\log_a b \text{ and } \frac{y}{x}=\frac{1}{\log_a b}=\log_b a.$$

Hence, if the logarithm of any number in the system in which the base is a be multiplied by $\frac{1}{\log_a b}$, we obtain the logarithm of the number in the system in which the base is b

The common logarithms have been calculated from the Napierian logarithms. Let l and L be the logarithms of the same number in the common and Napierian systems respectively, then

$$l=\frac{1}{\log_e 10}L,$$

$$\log_e 10=2\,30258509\dots\,.=2{\cdot}3026 \text{ approx.},$$

$$\text{and} \qquad \frac{1}{2{\cdot}30258509}=0{\cdot}43429448.\ \dots=0{\cdot}4343 \text{ approx.}$$

Hence, the common logarithm of a number may be obtained by multiplying the Napierian logarithm of the same number by 0·4343 ...

To convert common into Napierian logarithms multiply by 2·3026 instead of the more accurate number 2·30258509.

The preceding rules will be best understood by a careful study of a few examples.

Ex. 1. Log 10 to base e is 2·3026.

$$\log_e 10 = 2{\cdot}3026,$$

or
$$e^{2{\cdot}3026} = 10.$$

From this relation any number which is a power of 10 may be expressed as a power of e. Thus, log 19 5 = 1·29.

$$\therefore\ 19{\cdot}5 = 10^{1{\cdot}29} = e^{2{\cdot}3026 \times 1{\cdot}29} = e^{2{\cdot}9703},$$

or
$$\log_{10} 19{\cdot}5 = 1{\cdot}29,\ \log_e 19{\cdot}5 = 2{\cdot}9703.$$

Ex. 2. Find $\log_e 3$ and $\log_e 8{\cdot}43$.

$$\log_e 3 = 0{\cdot}4771 \times 2{\cdot}3026 = 1{\cdot}0986,$$
$$\log_{10} 8{\cdot}43 = 0{\cdot}9258;$$
$$\log_e 8{\cdot}43 = 0{\cdot}9258 \times 2{\cdot}3026 = 2{\cdot}1318.$$

Ex 3 Find log 13 to base 20.

Here log 13 = 1·1139, also log 20 = 1 3010.

$$\therefore\ \log_{20} 13 = \frac{1{\cdot}1139}{1{\cdot}3010} = 0{\cdot}8562.$$

Methods of computation.—Careful attention should be given to the *method* adopted in carrying out all computations These should in all cases be so arranged that any results obtained can be checked from time to time as the work proceeds Finally, where possible, any convenient rough check should be used to make sure that the result obtained is a reasonable one. In working with four-figure logarithms, the results obtained are only approximate; they give results true to three significant figures, the fourth figure although not necessarily accurate is usually not far wrong. When greater accuracy is required, five, six, or seven-figure logarithms should be used.

Ex. 4. Find the value of

$$3{\cdot}142^{1{\cdot}3} \times 0{\cdot}063 \times 10{\cdot}17^{-0{\cdot}09}.$$

Denoting the value required by x we have

$$x = 3{\cdot}142^{1{\cdot}3} \times 0{\cdot}063 \times 10{\cdot}17^{-0{\cdot}09};$$

$$\begin{aligned}\therefore\ \log x &= 1{\cdot}3 \log 3{\cdot}142 + \log 0{\cdot}063 - 0{\cdot}09 \log 10{\cdot}17 \\ &= 1{\cdot}3 \times 0{\cdot}4972 + \bar{2}{\cdot}7993 - 0{\cdot}09 \times 1{\cdot}0072 \\ &= 0{\cdot}64636 + \bar{2}{\cdot}7993 - 0{\cdot}09065 \\ &= \bar{1}{\cdot}3550 = \log 0{\cdot}2265;\end{aligned}$$

$$\therefore\ x = 0{\cdot}2265.$$

Ex. 5 The relation between Q, the quantity of water in cubic feet per second passing over a triangular gauge notch, and H, the height, in feet, of the surface of the water above the bottom of the notch, is given by $Q \propto H^{\frac{5}{2}}$.

When H is 1, Q is found to be 2·634. What is the value of Q when H is 4?

If the area of the reservoir supplying the notch is 80000 square feet, find the time in which a volume of water 80000 square feet in area and 3 inches in depth will be drawn off when H remains constant and equal to 4 ft.

The relation between Q and H may be written $Q = kH^{\frac{5}{2}}$, where k is a constant.

$$\text{When } H \text{ is } 1,\ Q = k \times 1;\ \therefore\ k = 2{\cdot}634.$$

$$\text{When } H \text{ is } 4,\ Q = 2{\cdot}634 \times 4^{\frac{5}{2}},$$

$$\text{or } \log Q = \log 2{\cdot}634 + \tfrac{5}{2}\log 4 = 1{\cdot}9259;$$

$$\therefore\ Q = 84{\cdot}31 \text{ cub. ft.}$$

$$\text{Volume of water} = \frac{80000 \times 3}{12} = 20000 \text{ cub. ft.}$$

$$\text{Time required} = \frac{20000}{84{\cdot}31 \times 60} = 3{\cdot}953 \text{ minutes.}$$

Ex. 6. If pv^k is constant; and if $p = 1$ when $v = 1$, find for what value of v, p is 0·2. Do this for the following values of k, 0·8, 0·9, 1·0, 1·1.

Let c denote the constant, then $pv^k = c$.

Substituting the simultaneous values $p = 1$, $v = 1$;

$$\therefore\ 1^k = c;\quad \therefore\ c = 1.$$

Thus when $p = 0{\cdot}2$ we have

$$0{\cdot}2v^k = 1;$$

$$v = 5^{\frac{1}{k}};$$

$$v = 5^{\frac{1{\cdot}0}{\cdot 8}} = 5^{1{\cdot}25}$$

$$\log v = 1{\cdot}25 \log 5 = 0{\cdot}8738;$$

$$\therefore\ v = 7{\cdot}476.$$

Similarly, when k has the values 0·9, 1·0, and 1·1, corresponding values of v are found to be 5·98, 5, and 4·32 respectively.

Ex. 7. In steam vessels of the same kind it is found that the relation between H, the horse power; V, the speed in knots; and D, the displacement in tons, is given by $H \propto v^3 D^{\frac{2}{3}}$.

Given $H = 35640$, $V = 23$, and $D = 23000$, find the probable numerical value of H when V is 24.

The relation may be written in the form

$$H = kV^3D^{\frac{2}{3}}, \text{ where } k \text{ is a constant.}$$

To find the value of k, substitute the given quantities

$$35640 = k \times (23)^3 \times (23000)^{\frac{2}{3}};$$

$$\therefore\ k = \frac{35640}{23^3 \times 23000^{\frac{2}{3}}}.$$

To find H when V is 24 we have

$$H = k \times 24^3 \times (23000)^{\frac{2}{3}}$$

$$= 35640 \times \left(\frac{24}{23}\right)^3 \times \left(\frac{23000}{23000}\right)^{\frac{2}{3}}$$

$$= 35640 \times \left(\frac{24}{23}\right)^3.$$

$$\log H = \log 35640 + 3(\log 24 - \log 23)$$

$$= 4{\cdot}5519 + 0{\cdot}0555 = 4{\cdot}6074 = \log 40500;$$

$$\therefore\ H = 40500$$

Ex. 8. In any class of turbine, if P is the power of the water, n the rate of revolution, H the height of the fall, and R the average radius at the place where water enters the wheel, then it is known that for all sizes

$$n \propto H^{1{\cdot}25}P^{-0{\cdot}5}, \qquad \text{(i)}$$

$$R \propto P^{0{\cdot}5}H^{-0{\cdot}75}. \qquad \text{(ii)}$$

In the list of a particular maker a turbine for a fall of 6 feet, 100 horse-power, 50 revolutions per minute, is 2·51 feet radius. By means of this n and R may be calculated for all the other turbines of the list. Find n and R for a fall of 20 feet and 75 horse-power.

Here $$n = kH^{1{\cdot}25}P^{-\frac{1}{2}}, \qquad \text{(iii)}$$
where k is a constant.

Substituting the given values,

$$50 = k \times (6)^{1{\cdot}25} \times (100)^{-\frac{1}{2}};$$

$$\therefore\ k = \frac{500}{6^{1{\cdot}25}}.$$

When H is 20, P is 75; to find n, we have, from (iii),

$$n = 500 \times \left(\frac{20}{6}\right)^{1{\cdot}25} \times (75)^{-\frac{1}{2}};$$

$$\therefore\ \log n = \log 500 + 1{\cdot}25(\log 20 - \log 6) - \tfrac{1}{2}\log 75$$

$$= 2{\cdot}4150;$$

$$\therefore\ n = 260.$$

In a similar manner, from (ii),

$$R = kP^{0.5}H^{-0.75}; \qquad \ldots\ldots\ldots\ldots \text{(iv)}$$

$$\therefore \quad k = \frac{2.51}{10} \times (6)^{0.75}$$

Substituting this value for k in (iv), we have, when H is 20 and P is 75,

$$R = 2.51 \times \left(\frac{6}{20}\right)^{0.75} \times \left(\frac{75}{100}\right)^{0.5}$$

$$= 2.51 \times (0.3)^{0.75} \times (0.75)^{0.5};$$

$$\therefore \log R = 0.3997 + \bar{1}.6078 + \bar{1}.9375 = \bar{1}.9450;$$

$$\therefore R = 0.881$$

Logarithms of trigonometrical ratios.—In Table IX. the sine, cosine, tangent, etc., for angles of a degrees from 0° to 90° are tabulated. In addition, by means of the numbers arranged in a horizontal direction, and by the columns of difference, the value of any of the above ratios can be obtained to the nearest minute. These ratios give the magnitude of all such angles with the conventions referred to in Chap. II. Having obtained the required number from the table, operations involving multiplication, division, involution, and evolution can be carried out in the usual manner.

Ex. 1. From Table IX. find the values of

$$\sin 161°, \quad \tan 127°, \quad \text{and} \quad \cos 104°.$$

As shown on p. 17, $\sin A = \sin(180° - A)$

Hence $\sin 161° = \sin(180° - 161°) = \sin 19°$,

and $\sin 19° = 0.3256 = \sin 161°$

$$\tan 127° = -\tan(180° - 127°) = -\tan 53°.$$

Hence, from Table IX, $\tan 127° = -1.3270$

Similarly, $\cos 104° = -\cos(180° - 104°) = -\cos 76°$;

$$\cos 104° = -0.2419$$

Ex. 2. Find the value of

$$\sin 161° \tan^2 127° \div \sqrt[3]{(\cos 104°)}.$$

Since $\cos 104° = -\cos 76°$,

this may be written as

$$x = -\sin 19° \tan^2 53° \div \sqrt[3]{(\cos 76°)}.$$

From Table IX. $\sin 19° = 0{\cdot}3256$,
$\tan 53° = 1{\cdot}3270$,
$\cos 76° = 0{\cdot}2419$.

Now $-x = \sin 19° \tan^2 53° \div \sqrt[3]{(\cos 76°)}$.

$$\therefore \log(-x) = \log 0{\cdot}3256 + 2\log 1{\cdot}3270 - \tfrac{1}{3}\log 0{\cdot}2419$$
$$= \bar{1}{\cdot}5127 + 2 \times 0{\cdot}1229 - \tfrac{1}{3}(\bar{1}{\cdot}3836)$$
$$= \bar{1}{\cdot}5127 + 0{\cdot}2458 - \bar{1}{\cdot}7945$$
$$= \bar{1}{\cdot}9640;$$
$$x = -\ 0{\cdot}9204.$$

Ex. 3. Find the values of

$$m = \frac{5400}{\pi} \log_e \frac{1 + \sin l}{1 - \sin l} \quad \ldots \quad \ldots \quad \text{(i)}$$

When $l = 0°, 35°, 65°$ . . (a), (b), (c)

(a) When $l = 0°$, $m = 0$.

(b) When $l = 35°$; $\sin 35° = 0\ 5736$.

Substituting in (i),

$$m = \frac{5400}{\pi} \log_e \frac{1{\cdot}5736}{0\ 4264}$$
$$= \frac{5400}{\pi}\left(0\ 1970 - \bar{1}{\cdot}6298\right)2{\cdot}303$$
$$= \frac{5400}{\pi} \times 0\ 5672 \times 2\ 303$$
$$= 2246.$$

(c) Similarly, when $l = 65°$

$$m = \frac{5400}{\pi} \log_e \frac{1\ 9063}{0\ 0937} = 5180.$$

Ex. 4. If $a = 5$, $b = 200$, $c = 600$, $g = -0{\cdot}1745$ radian, find the value of

$$ae^{-bt}\sin(ct + g) \quad . \quad \ldots \quad \text{(i)}$$

(a) When $t = 0{\cdot}001$.

(b) When $t = 0{\cdot}01$.

(c) When $t = 0{\cdot}1$.

Denoting the value of the given expression by y, and substituting the given values, we have

$$y = 5e^{-200t}\sin(600t - 0{\cdot}1745). \quad \ldots \quad \text{(ii)}$$

(a) When t is 0·001, we have, from (ii),

$$y=5e^{-0\cdot2}\sin(0\cdot6-0\cdot1745)=5e^{-0\cdot2}\sin(0\cdot4255).$$

From Table VII., or by multiplying 0·4255 by 57°·3, we find 0·4255 radians to be 24° 23′.

$$\therefore \log y=\log 5-0\cdot2\log e+\log\sin 24^\circ\ 23'$$
$$=0\cdot6990-0\cdot0869+\bar{1}\cdot6157=0\cdot2278=\log 1\cdot69;$$
$$\therefore y=1\cdot69.$$

(b) When t is 0·01, we have, from (ii),

$$y=5e^{-2}\sin(6-0\cdot1745)=-5e^{-2}\sin 26^\circ\ 12'.$$
$$\log(-y)=0\cdot6990-0\cdot8686+\bar{1}\cdot6449=\bar{1}\cdot4753=\log 0\cdot2987;$$
$$\therefore y=-0\cdot2987.$$

(c) When t is 0·1,

$$y=5e^{-20}\sin(60-0\cdot1745)=-5e^{-20}\sin 7^\circ\ 44';$$
$$\therefore \log(-y)=0\cdot6990-8\cdot686+\bar{1}\cdot1290=\bar{9}\cdot1420=\log 1\cdot387\times10^{-9},$$
$$y=-0\cdot1387\times10^{-10},\ \text{or},\ 0\cdot000000001387.$$

Ex. 5. Solve the equations,

$$\text{(i)}\ 7^x=3y,\quad \text{(ii)}\ 6^x=5y.$$

Dividing (i) by (ii), we have

$$\left(\frac{7}{6}\right)^x=\left(\frac{3}{5}\right)=0\cdot6;$$
$$\therefore x(\log 7-\log 6)=\log 0\cdot6,$$

or $$x(0\cdot8451-0\cdot7782)=\bar{1}\cdot7782,$$

or $$0\cdot0669x=\bar{1}\cdot7782=-0\cdot2218;$$

$$x=-\frac{2218}{669}=-3\cdot31.$$

Substituting this value in Eq. (ii), we have

$$5y=6^{-3\cdot31},$$
$$\therefore \log y=-3\cdot31\log 6-\log 5$$
$$=-3\cdot31\times0\cdot7782-0\cdot6990$$
$$=\bar{4}\cdot7251;$$
$$y=0\cdot000531.$$

Hence the values are $x=-3\cdot31$, $y=0\cdot000531$.

Some simple artifices.—When a given algebraic or other expression contains terms connected by the signs of addition and subtraction, the terms must be separately evaluated and afterwards added or subtracted as required.

By means of a few simple artifices it is sometimes possible to change such expressions into the form of products and quotients.

The artifices are not, however, of much value except in those cases where many examples of the same kind have to be evaluated.

Ex. 1. Calculate the value of the expression,

$$a^{\frac{3}{5}} \sin \theta (a^2 - b^2)^{-\frac{1}{2}},$$

when $a = 11{\cdot}78$, $b = 5{\cdot}67$, $\theta = 0{\cdot}4712$ radians

From Table IX. $0{\cdot}4712$ radians $= 27°$. Hence, if x denotes the value of the given expression

$$\begin{aligned} x &= (11{\cdot}78)^{\frac{3}{5}} \sin 27°(11{\cdot}78 + 5{\cdot}67)^{-\frac{1}{2}}(11{\cdot}78 - 5\ 67)^{-\frac{1}{2}} \\ &= (11{\cdot}78)^{\frac{3}{5}} \times 0{\cdot}454 \times (17{\cdot}45 \times 6{\cdot}11)^{-\frac{1}{2}}; \\ \log x &= \tfrac{3}{5} \log 11{\cdot}78 + \log 0{\cdot}454 - \tfrac{1}{2}(\log 17\ 45 + \log 6{\cdot}11) \\ &= 0\ 6427 + \bar{1}{\cdot}6571 - \tfrac{1}{2}(1{\cdot}2417 + 0{\cdot}7860) \\ &= 0{\cdot}2998 - 1{\cdot}0138 = \bar{1}{\cdot}2860 = \log 0{\cdot}1932; \\ \therefore\ x &= 0{\cdot}1932. \end{aligned}$$

Again, in dealing with quantities of the form $a^2 + b^2$, we may use $\tan \theta = \dfrac{b}{a}$; and, as $\tan \theta$ may have any value, the solution is always possible. Thus, if $\tan \theta = \dfrac{b}{a}$,

$$\begin{aligned} a^2 + b^2 &= a^2\left(1 + \frac{b^2}{a^2}\right) = a^2(1 + \tan^2 \theta) \\ &= a^2 \sec^2 \theta, \end{aligned}$$

a form adapted to logarithmic computation.

In a similar manner the fraction $\dfrac{a-b}{a+b}$ becomes $\tan\left(\dfrac{\pi}{4} - \theta\right)$.

Ex 2. Evaluate $a^{\frac{3}{5}} \sin\theta \, (a^2+b^2)^{-\frac{1}{2}}$,
when $a=11\cdot78$, $b=5\cdot67$, $\theta=0\cdot4712$ radians.

In Table IX., 0·4712 radians corresponds to 27° and $\sin 27°=0\cdot4540$.

Putting $$\tan\phi=\frac{b}{a}=\frac{5\cdot67}{11\cdot78}=0\cdot4812.$$

From Table IX., ϕ is found to be 25° 42′.

Now $$a^2+b^2=a^2\sec^2\phi=\frac{a^2}{\cos^2\phi},$$
and $\cos 25°\,42'=0\cdot9011$.

Hence, if x denotes the value of the given expression, we have

$$x=(11\cdot78)^{\frac{3}{5}}\sin 27°(a^2\div\cos^2\phi)^{-\frac{1}{2}}$$
$$=(11\cdot78)^{\frac{3}{5}}\times 0\cdot454\times\frac{0\cdot9011}{11\cdot78}.$$
$$\log x=\tfrac{3}{5}\log 11\cdot78+\log 0\cdot454+\log \cdot9011-\log 11\cdot78$$
$$=0\cdot6427+\bar{1}\cdot6571+\bar{1}\cdot9547-1\cdot0712$$
$$=\bar{1}\cdot1833;$$
$$\therefore\ x=0\cdot1525.$$

EXERCISES. VI.

Find the value of

1. $2\cdot625^{2\cdot5}\times 0\cdot0625\times 16\cdot06^{-0\cdot083}$. 2. $23\cdot07\times 0\cdot1354$, $2307\div 1\cdot354$.

3. How many ciphers are there between the decimal point and the first significant figure in $(0\cdot0504)^{10}$?

Evaluate

4. $\dfrac{(0\cdot07197)^{\frac{1}{8}}}{\sqrt[5]{27}}$. 5. (i) $\sqrt[5]{0\cdot02348}$; (ii) $\left(\dfrac{5}{7}\right)^{0\cdot1345}$.

6. Find without using tables the value of x for which
$$\log x=3\log 18-4\log 12.$$

7. Calculate the numerical value of
$$(0\cdot084)^{\frac{1}{5}}\div(0\cdot34)^3.$$

8. Evaluate $2\cdot307^{0\cdot65}-23\cdot07^{-1\cdot25}$

9. In the formula $L=(D+d)\left\{\dfrac{\pi}{2}+\theta+\dfrac{1}{\tan\theta}\right\}$,
given $$\sin\theta=\frac{D+d}{2c},$$
find the value of L when $c=20$ ft., $D=6$ ft., and $d=3$ ft.

10. The loss of energy E through friction of every pound of water flowing with velocity v through a straight circular pipe of length l ft. and diameter d ft. is given by $0.0007lv^2 \div d$.

Given $v=8.5$ ft. per sec., $l=3000$ ft., $d=6$ inches, find E.

11. Find the value of E from the formula

$$E=\frac{4}{3}\frac{wl^3}{\pi d \times a^4},$$

when $w=15$, $l=18.23$, $d=3$, $a=\frac{3}{8}$.

12. If $x=e^{\mu\theta}$, find x when $e=2.718$, $\mu=0.4$, $\theta=3.142$

Also find x when $\mu=0.7$ and $\theta=180°$.

Evaluate

13. $\sqrt{\dfrac{8^{\frac{1}{6}} \times 11^{\frac{1}{5}}}{\sqrt[4]{18} \times \sqrt[5]{9}}}$.

14. $\dfrac{(21.43)^2 \times 3.142 \times 0.0642}{1.236 \times \sqrt{0.004376}}$

15. From the equation

$$P=\frac{806300 \times t^{2.19}}{L \times D},$$

find P when $t=\frac{1}{2}$, $L=20$, $D=36$.

Also find the value of P when t^2 is used instead of the more accurate value $t^{2.19}$

16 The relation between p and v may be expressed by

(i) $pv=c$, (ii) $pv^{1.0646}=c$, (iii) $pv^{1.13}=c$.

If when p is 1.5, $v=1$, find p in each case when $v=3.5$

Also find in each case the value of v when p is 0.5.

17. If $w=144\{p_1(1+\log_e r)-r(p_3+10)\}$ and if $p_1=100$, $p_3=17$, find w when r is $1\frac{1}{2}$, 2, 3, 4.

18. Compute $2.307^{0.65}$ and $23.07^{-1.25}$

19. To what base would the numbers given in Table II. have logarithms double those actually given?

20 Find the square root of

$$\frac{\sqrt[3]{0.0125} \times \sqrt{31.15}}{0.00081}.$$

21. Evaluate $\dfrac{(7.25)^{\frac{1}{3}} \times 1.005}{(0.0874)^2}$.

22. Evaluate I from the formula

$$I=I_0\left(\frac{t_1^2}{t_0^2}\frac{W_0+W_1}{W_0}-1\right),$$

given $I_0=88.2$, $t_0=1.29$, $t_1=1.64$, $W_1=6.4$, $W_0=44.1$.

23. Find x and y from the equations

$$\log_{10} x^3 + \log_{10} y^2 = \bar{1}\cdot 4571,$$
$$\log_{10} x - \log_{10} y = 0\cdot 2300.$$

24. Find the value of one root of the equation

$$(4)^{2x} - 8(4)^x + 12 = 0.$$

25. Find to three decimal places a value of x which satisfies the equation $$5^{x+2} = 8^{2x-1}$$

26. Find $\log\left(\dfrac{64}{35}\right)^{\frac{5}{8}}$ and $\log \sqrt[5]{62\cdot 5}.$

Solve the equations

27. $$2^x = 9$$

28. $$x^5 = \frac{11\cdot 6 \times 0\cdot 4785}{0\cdot 0278}.$$

29. Evaluate E from the formula $E = \dfrac{Wl^3}{48 I \delta}$, given $W = 16$, $l = 20$, $I = \dfrac{\pi}{64}(0\cdot 373)^4$ and $\delta = 2\cdot 44 \div 25\cdot 4$.

30. Find the value of x correct to three places of decimals that satisfies the equation $7^x = 3^{x+1} \div 2^{x-2}$.

31. Solve the equation $105^x = 100.$

32. Find the logarithms of

$$\sqrt[5]{6}\ ; \quad \frac{2}{3}\sqrt[3]{14\cdot 4}\ ; \quad \frac{72}{125}\sqrt{270} \times \frac{3}{16}\sqrt[5]{625}.$$

33. Find the logarithms to base e of

$$\text{(i)}\ \frac{8}{\sqrt{27}}, \quad \text{(ii)}\ \frac{3e^2}{512}, \quad \text{(iii)}\ \sqrt{\frac{2}{3}} \times \sqrt[3]{\frac{9}{16}} \times \sqrt[4]{\frac{64}{27}}.$$

34 Prove that $7\log\dfrac{16}{15} + 5\log\dfrac{25}{24} + 3\log\dfrac{81}{80} = \log 2.$

35. Solve the equation

$$\left(\frac{1}{2}\right)^{x+4} = (25)^{3x+2}.$$

36. If $$Q = 1000\sqrt{\frac{D^5 H}{GL}},$$

and if $D = \frac{3}{4}$, $H = 0\cdot 4$, $L = 10$, $Q = 145$ be a set of simultaneous values, find D when

$$H = 2, \quad L = 5000, \quad Q = 465.$$

What is the value of the constant G?

37. Find the value of

$$\log_e \frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}, \text{ when } x=0{\cdot}62.$$

38. Evaluate

$$\frac{E\cos\left[pt-\tan^{-1}\dfrac{2p\pi\cos\beta}{n^2-p^2}\right]}{\sqrt{n^4+2n^2p^2\cos 2\beta+p^2}},$$

when $E=100,\ n=5,\ p=3,\ \beta=\frac{\pi}{6},\ t=1{\cdot}2$

39. Find V and v from the equations

$$V=\frac{(88{\cdot}51)(\sqrt{R}-0{\cdot}3)}{\sqrt{\dfrac{1}{\sin\theta}-\log_e\dfrac{1}{\sin\theta}-1{\cdot}6}}-0{\cdot}084(\sqrt{R}-0{\cdot}03),$$

$$v=60\sqrt{R\sin\theta}+120R^{\frac{2}{3}}\sin^{\frac{2}{3}}\theta\ ;$$

(i) when $R=8,\quad \theta=0{\cdot}02$;

(ii) when $R=2{\cdot}56,\ \theta=0{\cdot}144$.

40. Find to four significant figures the value of

$$\sin 116^\circ \tan^2 218^\circ \div \sqrt[3]{(\cos 102^\circ)}.$$

41. Some particulars of steam vessels are given. Assuming in each case the relation H.P $\propto V^3D^{\frac{2}{3}}$ to hold, where H.P. denotes the horse-power at a speed of V knots and displacement D in tons, find in each case the probable H.P. necessary to give a speed of 24 knots for same displacement

Name	H P	V	D
(i) Paris, - -	20000	20·25	15000
(ii) Teutonic, -	18000	19·50	13800
(iii) Campania, -	30000	22·10	19000
(iv) Kaiser, - -	28000	22·62	20000
(v) Oceanic, -	28000	20·50	28500
(vi) Deutschland,	35640	23·0	23000

42. When water pours over a triangular notch $Q \propto H^{\frac{5}{2}}$ (where Q denotes the number of cubic feet per sec., and H the height of the surface in feet), when H is 2, Q is 14·9, find the number of gallons per minute when H is 4.

43. Find the value of $10e^{-0{\cdot}7t}\sin(2\pi ft+0{\cdot}6)$ when f is 225 and t is 0·003.

44. Find the value of $a^p+b^q+c^r$ when $a=5$, $b=3$, $c=0{\cdot}042$, $p=2{\cdot}43$, $q=-0{\cdot}246$, $r=0{\cdot}476$

45. Evaluate $(x^2-y^2)z^{-\frac{4}{5}}\tan 40°$ when

$$x=50{\cdot}9,\ y=14{\cdot}8,\ z=29{\cdot}29$$

46 If p is the pressure and u the volume in cubic feet of 1 lb of steam, then from $pu^{1{\cdot}0646}=479$ find u when p is 150.

47. If
$$y=\log_e\frac{1+x+x^2}{1-2x+x^2}+2\sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}},$$
find the values of y which correspond to the following values of x:

$$x=0,\ x=0\ 4,\ x=1.$$

Assume that the given angle is acute.

48. Solve the equation

$$(2{\cdot}065)^{-0{\cdot}048x}=0{\cdot}826.$$

CHAPTER VI.

EQUATIONS.

Equations.—A statement that two arithmetical, or algebraical, expressions are equal is called an **equation.**

Identity.—When an equality exists between two quantities, and the **two expressions are equal for all values** of the quantities involved, such a statement is called an **identity,** thus

$$a(b+c)=ab+ac,$$
$$(a+x)^2=a^2+2ax+x^2,$$
$$(a+b)(a-b)=a^2-b^2,$$

are examples of **identities**

Equation.--An algebraic expression in which an equality or relation exists between certain known and unknown quantities, which is only true for certain values of the quantities involved, constitutes an **equation.** Known quantities may be indicated by the letters a, b, c, etc., and unknown quantities by the letters x, y, z.

An equation consists of two equal parts, one on the left, the other on the right of the sign of equality, and the equation will still be true when both sides are .

(i) Equally increased, or diminished ; which is the same in effect as taking a quantity from one side of an equation and placing it on the other with altered sign.

(ii) Equally multiplied, or divided ; this includes changing the signs of all the terms by multiplying both sides of the equation by -1.

Degree of an equation.—When a given equation expressed in its simplest form contains only the first power of one, or more, unknown quantities, it is called a **simple equation.** All such equations are said to be of the **first degree** or **linear equations.**

Similarly, if an equation contains the second power of an unknown quantity, it is called a **~~quadratic equation~~.** If it contains the third power it is called a **cubic** equation, etc

Solution of an equation.—The symbol $f(x)$ is used to denote any expression which involves a variable quantity x, and is read as a **function** *of* x.

If y stands for the value of such a function, then we may write $y=f(x)$; and by giving a series of numerical values to x, a corresponding series of values can be obtained for y.

Thus, $2x-16$, $2x^2-8x+6$, $x^3-3x^2-10x+24$,

may be called functions of x. The highest power of x in the first is one; it is two in the second, and three in the third Hence, these may be described as of the *first*, *second*, and *third* degree, respectively.

If a given equation be written in the form $f(x)=0$, and the substitution of any quantity a satisfies the equation, then $x-a$ is a **factor**, or, $x=a$ is a **root** of the equation Such an equation is said to be solved when all those values of x are found which when substituted in the expression makes it vanish or makes one side identical with the other. Again, if by giving two different values to x, results are obtained with different signs, the curve joining the plotted points would obviously intersect the axis of x at some intermediate point, that is to say at least one root of the given equation lies between the assigned values of x.

As a simple example let $f(x)=2x-16$; then, if y denotes the value of the function, $y=2x-16$.

Let $x=9$; then, $2x-16=18-16=2$.

Again, let $x=7$; then, $2x-16=14-16=-2$.

Hence, the root lies between these values.

By substituting $x=8$, it is found that this value satisfies the given equation; and therefore $x=8$ is the root required.

Ex. 1. $$\frac{3x}{4}+\frac{x}{2}+3=3x-4.$$

First subtract 3, and next subtract $3x$ from each side, and we obtain
$$\frac{3x}{4}+\frac{x}{2}-3x=-7.$$

Multiplying both sides of the equation by 4, then
$$3x+2x-12x=-28;$$
$$\therefore \quad -7x=-28;$$
$$x=\frac{-28}{-7}=4$$

To prove that this value of x satisfies the given equation, it is only necessary to substitute 4 for x, and each side is seen to be equal to 8

Instead of subtraction we may remove any term, or terms, from one side of an equation to the other; or, in other words, we may **transpose** a term, or terms, taking care to alter the sign, or signs, as in the case of the terms 3 and $3x$ in the preceding example Hence, for the solution of a given simple equation we may deduce the following rule

Transpose all the unknown quantities to the left and all the known quantities to the right-hand side of the equation. Simplify if necessary, and finally divide by the coefficient of the unknown quantity.

Some of the methods which may be used in the solutions of equations may be seen from the following examples:

Ex. 2 $$\frac{12}{x}+\frac{1}{12x}=\frac{29}{24}.$$

Multiply both sides by $24x$;
$$288+2=29x,$$
$$29x=290,$$
$$x=10.$$

Ex. 3. Solve $$\frac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9. \qquad \ldots \ldots \quad (1)$$

This is a typical example in which, if we multiply out and afterwards proceed to square, troublesome expressions result.

But if $$\frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a+b}{a-b}=\frac{c+d}{c-d};$$

i.e. **we may add the numerator to the denominator to obtain a new numerator and subtract to obtain a new denominator.** This must be done on both sides of the equation. Thus in (i)

$$\frac{2\sqrt{4x+1}}{2\sqrt{4x}}=\frac{9+1}{9-1}=\frac{10}{8};$$

$$\therefore\ \frac{\sqrt{4x+1}}{\sqrt{4x}}=\frac{5}{4}.$$

Squaring both sides, $$\frac{4x+1}{4x}=\frac{25}{16};$$

$$\therefore\ 64x+16=100x;$$

$$x=\frac{16}{36}=\frac{4}{9}.$$

Fractional equations.—In the solution of equations involving fractions it is in many cases advisable to commence by clearing of fractions. This may be effected by multiplying by the L.C.M. of the denominators In some cases each side of a given equation may be simplified as in the following example.

Ex. 4. Solve $$\frac{x-15}{x-16}-\frac{x-4}{x-5}=\frac{x-6}{x-7}-\frac{x+5}{x+4}.$$

This may be written in the form

$$1+\frac{1}{x-16}-\left(1+\frac{1}{x-5}\right)=1+\frac{1}{x-7}-\left(1+\frac{1}{x+4}\right),$$

or $$\frac{1}{x-16}-\frac{1}{x-5}=\frac{1}{x-7}-\frac{1}{x+4};$$

$$\therefore\ \frac{x-5-x+16}{(x-5)(x-16)}=\frac{x+4-x+7}{(x+4)(x-7)},$$

or $$\frac{11}{(x-5)(x-16)}=\frac{11}{(x+4)(x-7)},$$

or $$(x+4)(x-7)=(x-5)(x-16),$$

$$x^2-3x-28=x^2-21x+80;$$

$$\therefore\ 18x=108,\ \therefore\ x=6.$$

Ex. 5. $$\sqrt{x-9}=\sqrt{x}-1.$$

Square both sides. $$x-9=x-2\sqrt{x}+1;$$

$$\therefore\ 2\sqrt{x}=10.$$

Divide by 2; $$\therefore\ \sqrt{x}=5.$$

Hence, $$x=25.$$

Ex. 6. Solve $\sqrt{4a+x}-\sqrt{x}=2\sqrt{b+x}$.

Square both sides.

$$4a+2x-2\sqrt{4ax+x^2}=4(b+x).$$

Transpose and divide by 2;

$$\therefore \sqrt{4ax+x^2}=x-2(a-b).$$

Square both sides.

$$4ax+x^2=x^2-4x(a-b)+4(a-b)^2;$$
$$x(8a-4b)=4(a-b)^2,$$
$$\text{or } 4x(2a-b)=4(a-b)^2;$$
$$\therefore x=\frac{(a-b)^2}{2a-b}.$$

In the preceding, and in all cases where the solution of an equation is obtained by the processes of involution or evolution, it is necessary to test whether the value obtained satisfies the given equation

EXERCISES VII.

Solve the equations:

1. $\frac{2x}{15}+\frac{x-6}{12}-\frac{3x}{20}=1\frac{1}{2}$.

2. $\frac{1}{2}(x-1)-\frac{1}{3}(2-x)+\frac{1}{4}(x+1)=x$

3. $\frac{5}{2}-\frac{x+4}{11}=x+\frac{1}{2}$

4. $\frac{x+3}{4}-\frac{x-3}{5}=\frac{x-5}{2}-2$.

5. $\frac{2x-5}{4}-\frac{x-2}{6}=\frac{x}{7}-\frac{1}{4}$.

6. $\frac{3x+5}{7}-\frac{6x+5}{9}=x-\frac{2}{3}$.

7. $\frac{x+2}{3}+2=\frac{x+4}{5}+\frac{x+6}{7}$.

8. $\frac{3x-13}{8}-\frac{4x+6}{9}=1-\frac{x-1}{10}$.

9. $\frac{3x-1}{3}+\frac{5}{12}=\frac{x}{4}+\frac{2x+1}{5}$

10. $\frac{1}{(x-1)(x-2)}+\frac{1}{(x+1)(x+2)}=\frac{2}{(x-1)(x+2)}$.

11. $\frac{x-\frac{1}{2}}{x-1}-\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right)=\frac{23}{10(x-1)}$.

12. $\frac{ax}{b}-\frac{1}{b}\left(\frac{1}{c}+x\right)+d=\frac{d}{b}\left(bx-\frac{1}{cd}\right)-\frac{x}{b}+\frac{a}{b}$.

13. $\frac{3}{7}(6x-7)+\frac{1-7x}{6}=x$.

14. $\frac{2x+1}{3}-\frac{3x-2}{4}=\frac{x-2}{6}$.

Solve the equations:

15. $11(x-5)-5(x-11)=5\frac{1}{4}$.

16. $0{\cdot}1x+\dfrac{0{\cdot}05x-0{\cdot}08}{0{\cdot}3}=0{\cdot}88-\dfrac{0{\cdot}03x-0{\cdot}08}{0{\cdot}5}$

17. $\dfrac{x+a}{x-c}=\dfrac{a+c}{a-c}$.

18. $\sqrt{x+7}=\sqrt{x}+1$.

19. $\dfrac{a}{bx}+\dfrac{b}{ax}=a^2+b^2$.

20. $\dfrac{a-x}{bc}+\dfrac{b-x}{ac}+\dfrac{c-x}{ab}=0$.

21. $\dfrac{(x+a)^3-(x-a)^3}{2a}=3(x+a)^2-5a^2$.

22. $\dfrac{xa}{b}+\dfrac{xb}{a}=a^2+b^2$.

23. $\dfrac{1}{a(x-b)}+\dfrac{1}{b(x-c)}=\dfrac{1}{a(x-c)}$.

24. $\dfrac{5x-9}{\sqrt{5x}-3}=x+3$.

25. $\sqrt{(x+4)}+\sqrt{(2x+10)}=\sqrt{2}$.

26. $\sqrt{x}+\sqrt{x+3}=\dfrac{5}{\sqrt{x+3}}$

27. $(x+3)^3-3(x+2)^3+3(x+1)^3-x^3=x+3$

Problems producing equations.—When told in words how to deal arithmetically with a given quantity, it is of importance to be able to state the matter algebraically. The true meaning of such a question, or problem, must in the first place be perfectly understood and its conditions exhibited by algebraical symbols in the clearest manner possible The following are a few typical examples of problems of this kind

Ex. 1. Twice a certain number exceeds four-fifths of its half by 40. Find the number.

Let x denote the number; then, twice the number is $2x$. Also four-fifths of its half is $\frac{4}{5}\times\frac{x}{2}$

Hence, by the question

$$2x-\frac{4}{5}\times\frac{x}{2}=40;$$

$$20x-4x=400,$$

$$\text{or } 16x=400;$$

$$x=\frac{400}{16}=25$$

Substituting this value the equation is satisfied.

Ex. 2. The total length of 4 pieces of copper wire is 50 feet; the second is twice, the third three times, and the fourth is four times as long as the first. Find the length of each piece.

If x denotes the number of feet in the first,

then $2x$	,,	,,	,,	second,
$3x$	,,	,,	,,	third,
and $4x$	,,	,,	,,	fourth;

$$\therefore \quad x + 2x + 3x + 4x = 50,$$

$$10x = 50; \qquad x = 5 \text{ ft}$$

The lengths are 5, 10, 15 and 20 ft respectively

Ex 3. In ascending a mountain, a man took half as long again to climb the second third as he did to climb the first third, and a quarter as long again for the last third as for the second third; he took altogether 5 hours 50 minutes. Find the time he spent on the first third of the journey.

If x denotes the time taken for the first third,

then $\frac{3}{2}x$	,,	,,	,,	second third,
and $\frac{5}{4} \times \frac{3}{2}x$	,,	,,	,,	last third.

Also 5 hours 50 minutes = 350 minutes.

Hence $$x + \frac{3}{2}x + \frac{15}{8}x = 350:$$

$$\therefore \quad 35x = 8 \times 350,$$

$$x = 80 \text{ minutes}$$

The time spent on the first third = 1 hour 20 minutes.

Ex. 4. The sides of a triangle ABC are together 61 miles long; BC is $\frac{5}{6}$th of AB and 3 miles longer than CA. Find the lengths of the sides severally.

Let x denote the length of AB

Then $\frac{5}{6}x$ will denote the length of BC,

and $\frac{5}{6}x - 3$,, ,, AC.

Hence $$x + \frac{5}{6}x + \frac{5}{6}x - 3 = 61;$$

$$16x = 384, \text{ or } x = 24.$$

Also $$\frac{5}{6}x = 20, \text{ and } \frac{5}{6}x - 3 = 17.$$

The three sides are 24, 20 and 17 respectively

Ex. 5. The perimeter of a triangle is 22 feet, the base is 3 feet longer than one side, and 5 feet longer than the other. Find the lengths of the sides.

Let x denote the length of the base. Then $x-3$ and $x-5$ are the lengths of the sides.

$$\therefore x+x-3+x-5=22,$$

$$3x=30, \quad x=10,$$

and the sides are 7 and 5.

EXERCISES. VIII.

1. A person is walking with uniform speed, and when he has completed half his journey he increases his pace in the ratio of 3 to 2, and arrives at his destination 40 minutes earlier than he would otherwise have done. How long was he walking the first half?

2. A and B distribute £60 each among a certain number of persons. A relieves 40 persons more than B does, and B gives to each person 5 shillings more than A. How many persons did A and B relieve?

3. Two cyclists, A and B, ride a mile race. In the first heat A wins by 6 seconds. In the second heat A gives B a start of $58\frac{2}{3}$ yards and wins by 1 second. Find the rates of A and B in miles per hour.

4. At present B's age is to A's in the ratio of 4 to 3; but fifteen years ago it was in the ratio of 3 to 2. Find their ages.

5. Divide £490 among A, B and C, so that B shall have £2 more than A, and C as many times B's share as there are shillings in A's share.

6. I have thought of a number; I multiply it by $2\frac{1}{3}$ and add 7 to the product; I then multiply the result by 8 times the number thought of; next I divide by 14 and subtract from the quotient 4 times the number thought of; I thus obtain 2352. What number did I think of?

7. A distributes £180 in equal sums amongst a certain number of people. B distributes the same sum in equal portions amongst 40 persons fewer, but gives to each person £6 more than A does. How much does A give to each person?

8. A traveller starts from A towards B at 12 o'clock and another starts at the same time from B towards A. They meet at 2 o'clock, at 24 miles from A; and the one arrives at A while the other is still 20 miles from B. What is the distance between A and B?

9. A man walks a certain distance in 4 hours If he were to reduce his rate by one-sixteenth he would walk one mile less in that time. What is his rate?

10. If one part of £400 is put out at 4 per cent and the other part at 5 per cent., and if the yearly income be £18. 5*s*., what are the parts?

11. A sum of money amounts to £546 in three years at simple interest, and to £726 in 7 years Find the sum and the rate per cent.

12. A sum of £23 14*s*. is divided between A, B and C. If B gets 20 per cent. more than A, and 25 per cent more than C, how much does each get?

13. A man spends £1000 of his capital, and then spends $\frac{2}{5}$ of the remainder; then after receiving a legacy of £100 he has half his original capital. Find its amount

14. A person has £1750 invested so as to bring in an annual income of £77; part is lent on a mortgage at 4 per cent., the rest on loan at 5 per cent. How much is in the mortgage?

15 Show that the square of the sum of any two consecutive numbers is greater by 1 than four times the product of the numbers.

16. Show that the cube of the sum of any two numbers is equal to the sum of their cubes together with three times their product multiplied by their sum.

Simultaneous equations.—Equations containing two or more unknown quantities are called **simultaneous** equations The simplest case occurs when each of two given equations contains the first power only of the two unknown quantities; in such an equation, if values of one variable are assumed, then corresponding values of the other can be calculated. When there are two distinct and independent equations, only one pair of values will simultaneously satisfy both equations. Equations of this kind which are to be satisfied by the same pair of values of x and y are called **simultaneous equations.**

Ex. 1. $$2x+5y=48, \quad\ldots\ldots\ldots\ldots\ldots (1)$$

This may be written in the form $y=\frac{48-2x}{5}$; and if we substitute values 0, 1, 2 for x, corresponding values of y can be calculated and the assemblage of plotted points will lie in a straight line.

If, in addition to (i), we have the equation,

$$3x+4y=44, \qquad \ldots\ldots \text{(ii)}$$

then the equations (i) and (ii) form a pair of simultaneous equations, and the process of solving them simply consists in finding those simultaneous values of the variables x and y which will satisfy the given equations.

First method.—Three methods may be used, the first, which should always be used, being the most important. (a) By multiplication, or division, the coefficients of x, or y, are made the same in both equations. Then, by addition, or subtraction, an equation involving only one unknown quantity is obtained, and this may be solved in the usual manner.

Thus, multiplying Eq (i) by 4 and Eq. (ii) by 5,

$$15x+20y=220. \qquad \ldots\ldots \text{(iii)}$$

$$8x+20y=192$$

By subtraction $\quad 7x = 28;$

$$x=\frac{28}{7}=4.$$

Substitute this value of x in (i) and we get

$$5y=48-2x=40;$$

$$\therefore \; y=\frac{40}{5}=8$$

Hence, the pair of values $x=4$, $y=8$, satisfies the given equations. This result should be verified by substituting the values obtained in the given equations.

Second method.—The values of x and y may be obtained by substitution

Thus, given $2x+5y=48$ (i), $3x+4y=44$ (ii).

From (i), $y=\frac{48-2x}{5}$

Substituting this value in (ii),

$$3x+4\frac{(48-2x)}{5}=44.$$

Multiply both sides by 5 ;

$$\therefore\ 15x+192-8x=220\ ;$$

$$7x=220-192=28,$$

$$\text{or } x=4.$$

Substitute this value of x in (i) or (ii), then y is found to be 8.

Third method.—From each of the two given equations a value for y in terms of x may be obtained. Then, by equating the two values so obtained, another equation is obtained involving only x, and this may be solved in the manner shown for equations of one variable.

Ex. 2. Solve $3x-\frac{y}{2}=5, \quad \ldots \quad \ldots \quad$ (i)

$\frac{x}{3}+\frac{y}{4}=3 \quad \ldots \quad \ldots \quad$ (ii)

From (i) $\frac{y}{2}=3x-5$; $\quad y=6x-10 \quad \ldots \quad$ (iii)

From (ii) $\frac{y}{4}=3-\frac{x}{3}$; $\quad y=12-\frac{4x}{3}. \quad \ldots \quad$ (iv)

Equating (iii) and (iv), we have

$$6x-10=12-\frac{4x}{3}\ ;$$

$$6x+\frac{4x}{3}=22,$$

or

$$22x=66\ ;$$

$$\therefore\ x=3.$$

Substituting this value for x in (iii) or (iv), we obtain $y=8$.

Elimination.—From two distinct and independent equations containing two unknown quantities, one unknown can be eliminated by the processes just referred to, the resulting equation will then consist of an unknown and a known quantity, and its solution can be effected in the usual manner.

Similarly, three equations containing three unknown quantities may be reduced to two equations containing two unknowns. Then the two can be reduced to one equation containing only one unknown ; and from this, the value of that unknown quantity is obtained and the remaining two found by substitution.

Ex. 3. Solve the simultaneous equations,

$$2x+4y=20, \quad \ldots\ldots \quad \text{(i)}$$

$$3x+2y=18. \quad \ldots\ldots \quad \text{(ii)}$$

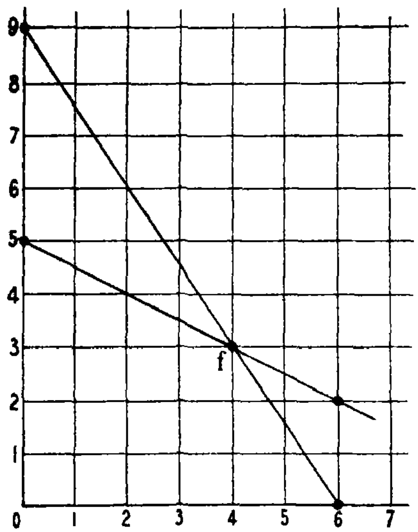

FIG. 14.—Solution of simultaneous equations.

From (i) $y=-\frac{1}{2}x+5$.

When

$x=6,\ y=-3+5=2$.

When

$x=0,\ y=5$.

By plotting these values the line (i) is obtained.

Similarly, from (ii), when

$x=6,\ y=0$;

and $x=0,\ y=9$.

By plotting these values the lines can be drawn through the plotted points; f the point of intersection of the two lines (Fig. 14) is a point common to both lines and the co-ordinates of point f, $x=4$ and $y=3$ are the values which satisfy the given equations.

Ex. 4 Solve the equations,

$$2x+3y=13 \quad \ldots\ldots \quad \text{(i)}$$

$$2x+3y=17. \quad \ldots\ldots \quad \text{(ii)}$$

These form two distinct equations; but, assuming a series of values 0, 1, 2, etc., for x, and calculating corresponding values of y, it will be found that none of the values obtained from (i) coincide with those from (ii). In other words, simultaneous values of x and y satisfying the two equations cannot be obtained. On plotting, it is seen that the two lines are parallel.

Thus (i) may be written $y=\dfrac{13-2x}{3}$.

When $x=2,\ y=3$; and when $x=5,\ y=1$.

The line passing through the points $x=2$, $y=3$, and $x=5$, $y=1$, or (2, 3) (5, 1) is shown at ab (Fig. 15).

From (ii),

$$y=\frac{17-2x}{3}.$$

When

$$x=1,\quad y=5;$$

and when

$$x=7,\quad y=1.$$

The line is indicated by cd (Fig. 15).

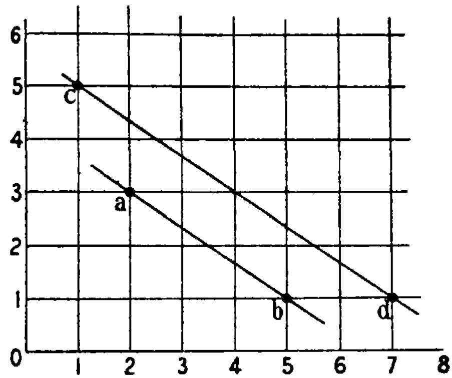

FIG 15 – Parallel lines

Some of the artifices which may be usefully employed in the solution of equations may be seen from the following examples.

Ex. 5
$$x+y=c, \quad \ldots \quad \text{(i)}$$
$$ax=by. \quad \ldots \quad \text{(ii)}$$

Multiply (i) by b and add to (ii)

$$x(a+b)=bc;$$

$$\therefore\ x=\frac{bc}{a+b}.$$

From (i),
$$y=c-x=c-\frac{bc}{a+b}$$

$$=\frac{ac}{a+b}.$$

Ex. 6
$$x+2y+3z=17, \quad \ldots \quad \text{(i)}$$
$$2x+3y+\ z=12, \quad \ldots \quad \text{(ii)}$$
$$3x+\ y+2z=13. \quad \ldots \quad \text{(iii)}$$

Multiply (i) by 2 and subtract Eq (ii) from it;

$$\begin{array}{r} 2x+4y+6z=34 \\ \underline{2x+3y+\ z=12} \\ y+5z=22 \end{array} \quad \ldots \quad \text{(iv)}$$

Multiply (ii) by 3 and (iii) by 2 and subtract;

$$\therefore\ \begin{array}{r} 6x+9y+3z=36 \\ \underline{6x+2y+4z=26} \\ 7y-\ z=10 \end{array} \quad \ldots \quad \text{(v)}$$

Multiply (v) by 5 and add to (iv);

$$\therefore \begin{aligned} 35y - 5z &= 50 \\ y + 5z &= 22 \\ \hline 36y &= 72, \\ y &= 2. \end{aligned}$$

From (iv), $z = \dfrac{22-2}{5} = 4$;

and from (i), $x = 17 - 4 - 12 = 1$.

Ex. 7. Solve $\dfrac{x}{b+c-a} = \dfrac{y}{c+a-b} = \dfrac{z}{a+b-c}$, ... (i)

$x + y + z = n$. . . (ii)

From (i), $\dfrac{x}{b+c-a} = \text{etc} = \dfrac{x+y+z}{a+b+c} = \dfrac{n}{a+b+c}$ from (ii);

$$x = \frac{n(b+c-a)}{a+b+c},$$

$$y = \frac{n(c+a-b)}{a+b+c},$$

$$z = \frac{n(a+b-c)}{a+b+c}$$

In many cases it is more convenient to solve for $\dfrac{1}{x}$ and $\dfrac{1}{y}$... instead of x, y.

Ex. 8. Solve $\dfrac{5}{x} + \dfrac{7}{y} = 2$, ... (i)

$5x + y = xy$. (ii)

Divide both sides of equation (ii) by xy.

$\dfrac{1}{x} + \dfrac{5}{y} = 1$ . . . . (iii)

Multiply (iii) by 5 and subtract (i) from it;

$$\therefore \frac{18}{y} = 3,$$

giving $y = 6$.

Substitute this value in (i).

$$\frac{5}{x} = 2 - \frac{7}{6} = \frac{5}{6};$$

$$\therefore x = 6.$$

It is better to keep the fractional form. The attempt to clear the equations from fractions would introduce a new term xy.

EXERCISES. IX.

Solve the equations ·

1. $\dfrac{x-3}{5}=\dfrac{y-7}{2}$; $11x=13y$

2. $3x-2y=2x+3y=26$.

3. $\dfrac{x+4}{7}-\dfrac{x-y-1}{4}=2x-4,$
$2y-4-\dfrac{3x-2y}{3}=3x.$

4. $x+\frac{1}{2}y=4,$
$y+2z=12,$
$\frac{3}{5}z-\frac{2}{3}x=1.$

5. $12x+11y=12,$
$42x+22y=40{\cdot}5.$

6. $\dfrac{x}{3}+5=\dfrac{2y}{3},$
$y-x=\dfrac{x}{3}.$

7. $2x+\dfrac{y}{3}=x+12,$
$y-x+20=\dfrac{x+40}{2}$

8. $\left.\begin{array}{l} 7x-4y=b-a, \\ 8y+21x=5p-3a-2b \end{array}\right\}$

9. $\frac{5}{2}x-\frac{2}{3}y=9=\frac{5}{3}y-x.$

10. $2x+3y=13$, $5x-3y=1$

11. $x+2y=3$, $2x-3y=3$.

12. $\left.\begin{array}{l} 3x+2y+5z=1, \\ 5x+3y-2z=2, \\ 2x-5y-3z=7 \end{array}\right\}$

13. $\left.\begin{array}{r} 3{\cdot}4x-0{\cdot}02y=0{\cdot}01, \\ x+0{\cdot}2y=0{\cdot}6 \end{array}\right\}$

14. $y=\dfrac{x}{m}+am$, $y-2am=-m(x-am^2)$.

15. $ax-by=2ab$, $2bx+2ay=3b^2-a^2$.

16. $\left.\begin{array}{r} x+y=a+b, \\ bx+ay=2ab \end{array}\right\}$

17. $\left.\begin{array}{l} \dfrac{a}{x}+\dfrac{b}{y}=cd, \\ \dfrac{b}{x}-\dfrac{a}{y}=ef \end{array}\right\}$

18. $\left.\begin{array}{l} \dfrac{x-a}{b-a}=\dfrac{y+b}{a+b}, \\ \dfrac{x+a}{a-b}=\dfrac{y-b}{a+b}. \end{array}\right\}$

19. $x+y+z=6$, $2x+y-z=1$, $3x-y+z=4$.

20. $\dfrac{x}{a}+\dfrac{y}{b}=1$, $\dfrac{x}{3a}+\dfrac{y}{6b}=\dfrac{2}{3}$.

21. $\left.\begin{array}{l} x-y+z=n, \\ \dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c} \end{array}\right\}$

22. $\left.\begin{array}{l} y+z=x+4a, \\ z+x=y+2a, \\ x+y=z. \end{array}\right\}$

23. Solve the simultaneous equations:

$$\text{(i) } y^2 = px,\ y = mx + \frac{p}{4m};$$

$$\text{(ii) } \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=1.$$

24 From the relation $y=\dfrac{3x^2-10x+9}{5x^2-16x+14}$,

prove that y is never greater than $\frac{2}{3}$ or less than $\frac{1}{2}$, for real values of x.

Problems producing simultaneous equations.—In preceding examples the conditions of a given problem have been expressed in terms of one unknown quantity x. It is, however, much easier in many problems, and indeed indispensable in others, to use two or more unknown quantities These are usually expressed by the letters x, y, z, ... In such equations it is necessary to obtain as many independent equations as there are unknown quantities involved. From these the solution is effected either by elimination or by substitution

Ex. 1. If 9 horses and 7 cows sell for £300 and 6 horses and 13 cows sell for the same amount, what is the price of each?

(*a*) Let x denote the price of a horse, then $300 - 9x$ is the price of 7 cows.

$$\therefore\ \frac{300-9x}{7} \text{ is the price of each cow.}$$

Also, in the second case, $\dfrac{300-6x}{13}$ is the price of each cow.

$$\therefore\ \frac{300-9x}{7}=\frac{300-6x}{13};$$

$$\therefore\ x = £24, \text{ and } \frac{300-9x}{7} = £12.$$

(*b*) Let x denote the price of a horse and y the price of a cow.

Then $9x + 7y = 300.$ .... (i)

Also $6x + 13y = 300.$ .... (ii)

Multiply (i) by 2 and (ii) by 3 and subtract;

$$\begin{aligned} 18x + 39y &= 900 \\ 18x + 14y &= 600; \\ \hline 25y &= 300 \\ y &= £12. \end{aligned}$$

And by substitution in (i), $x = £24.$

Ex. 2. A number consisting of three digits (those in the tens' and hundreds' places being equal) is 49 times the sum of its digits. If the order of the digits be reversed, the number so formed will be less than the original number by 297. Find the original number.

Let x, y and z denote the three digits. Then, the number required is represented by $100x+10y+z$. Also the sum of the digits is $x+y+z$.

Hence, $$100x+10y+z=49(x+y+z). \quad \ldots \quad \text{(i)}$$

The number reversed would be $100z+10y+x$;

$$\therefore (100x+10y+z)-(100z+10y+x)=297. \quad \ldots \quad \text{(ii)}$$

Also, as the digits in the tens' and hundreds' places are equal,

$$x=y. \quad \ldots \quad \text{(iii)}$$

Substituting from (iii) in (i),

$$12x=48z, \text{ or } x=4z. \quad \ldots \quad \text{(iv)}$$

Also, from (ii), $$x-z=3;$$

$$x=z+3.$$

Substituting this value in (iv), and we find

$$12(z+3)=48z;$$

$$\therefore 36z=36, \text{ or } z=1.$$

Hence, from (iv), $$x=4=y,$$

and the number required is 441.

Ex. 3. If 3 thalers exceed 11 francs and 59 francs exceed 16 thalers, the excess in each case being a halfpenny, find the English equivalents of the thaler and the franc

Let x denote the value of a thaler and y the value of a franc

Then, from the first condition,

$$3x-11y=\tfrac{1}{2}. \quad \ldots \quad \text{(i)}$$

Also $$-16x+59y=\tfrac{1}{2}. \quad \ldots \quad \text{(ii)}$$

Multiplying (i) by 16 and (ii) by 3 and adding,

$$y=9{\cdot}5.$$

Substituting in (i), $$3x=\tfrac{1}{2}+(11\times 9{\cdot}5)=105;$$

$$\therefore x=35.$$

Hence, the value of a thaler is $35d.$, and of a franc is $9\tfrac{1}{2}d.$

Ex. 4. The receipts of a railway company are apportioned as follows: 49 per cent. for working expenses, 10 per cent. for the reserved fund, a guaranteed dividend of 5 per cent. on one-fifth of the capital, and the remainder, £40,000, for division amongst

the holders of the rest of the stock, being a dividend at the rate of 4 per cent. per annum. Find the capital and the receipts.

Let C denote the capital and R the receipts; 41 %, or $0{\cdot}41R$, is available for dividend. Of this $\frac{1}{20}$ of $\frac{C}{5}$, or $0{\cdot}01C$, goes to pay guaranteed dividend;

$\therefore$ $0{\cdot}41R - 0{\cdot}01C$ remains for ordinary dividend;

$$\therefore\ 0{\cdot}41R - 0{\cdot}01C = 40000. \qquad \text{(i)}$$

Also
$$0{\cdot}8C = 25 \times 40000;$$
$$C = £1{,}250{,}000$$

Substituting in (i),
$$0{\cdot}41R - 12500 = 40000;$$
$$R = \frac{5250000}{41} = £128048.\ 15s.\ 7d.$$

When the data of a problem furnishes only one equation involving two unknown quantities, the ratio between the two may in some cases be obtained.

Ex. 5. An alloy of copper, zinc, and tin contains 91 per cent. of copper, 6 of zinc, and 3 of tin. A second alloy containing copper and tin only is fused with the first, and the resulting alloy is found to contain 88 per cent. of copper, 4·875 of zinc, and 7·125 of tin. Find the proportion of copper and tin in the second alloy.

We may assume that in order to form the resulting alloy x parts of the second alloy are fused with 100 parts of the first. Then, as there is no zinc in the second alloy, we have the relation,

$$6 = \frac{4{\cdot}875}{100}(100 + x);$$
$$4{\cdot}875x = 600 - 487{\cdot}5 = 112{\cdot}5;$$
$$\therefore\ x = \frac{112500}{4875} = \frac{300}{13}.$$

Thus, in the resulting $\frac{1600}{13}$ parts of new alloy we have $\frac{88}{100} \times \frac{1600}{13}$ parts of copper.

Hence $\left(88 \times \frac{16}{13} - 91\right)$ parts of copper come from second alloy, and in like manner $\left(7{\cdot}125 \times \frac{16}{13} - 3\right)$ parts of tin come from second alloy;

therefore proportion is $\dfrac{\left(88 \times \frac{16}{13} - 91\right)}{7{\cdot}125 \times \frac{16}{13} - 3} = \frac{225}{13} \div \frac{75}{13} = \frac{3}{1}.$

Ex. 6. The total increase in the number of undergraduates of a certain university in a recent year over the number in the preceding year was $2\frac{1}{2}$ per cent. In the number of resident undergraduates there was an increase of 4 per cent., and in the number of non-resident undergraduates a decrease of 11 per cent. Find the ratio of the number of non-resident to the number of resident undergraduates.

Let x denote the number of resident undergraduates, and y the number of non-resident in the latter year, then we have, considering the ratio in the former year,

$$\frac{100}{104}x+\frac{100}{89}y=\frac{100}{102 \cdot 5}(x+y),$$

or
$$89\times 1025x+104\times 1025y=89\times 1040(x+y);$$
$$89x=936y,$$
$$\frac{x}{y}=\frac{936}{89}.$$

Ex. 7. The perimeter of a right-angled triangle is six times as long as the shortest side. Find the ratio of the two perpendicular sides.

Let c denote the hypotenuse, a the shortest side, and b the remaining side.

Then $\quad a+b+c=6a$, or $b+c=5a$. .. ..(1)

Also $\quad a^2+b^2=c^2$.

Hence, substituting from (1),

$$c^2=(5a-b)^2$$
$$=25a^2-10ab+b^2;$$
$$\therefore\ a^2+b^2=c^2=25a^2-10ab+b^2,$$

or
$$24a^2=10ab;$$
$$\frac{a}{b}=\frac{5}{12}.$$

Ex. 8. An examiner has marked a set of papers; the highest number of marks is 185, the lowest 42. He desires to change all his marks according to a linear law converting the highest number of marks into 250 and the lowest into 100; show how he may do this, and state the converted marks for papers already marked 60, 100, 150.

Let $y=ax+b$ denote the linear law, where y denotes the

number of marks on the new system, and x denotes the number of marks on the old system.

Then, substituting the given values, we have

$$250 = 185a + b \qquad \text{(i)}$$

$$100 = 42a + b \qquad \text{(ii)}$$

Subtracting, $\quad 150 = 143a$;

$$\therefore \; a = \frac{150}{143};$$

and from (ii), $\quad b = 100 - \frac{42 \times 150}{143} = \frac{8000}{143}.$

Hence, if y_1, y_2 and y_3 denote the respective number of marks,

then $$y_1 = \frac{150}{143} \times 60 + \frac{8000}{143} = 118 \cdot 9,$$

$$y_2 = \frac{150}{143} \times 100 + \frac{8000}{143} = 160 \cdot 8,$$

$$y_3 = \frac{150}{143} \times 150 + \frac{8000}{143} = 213 \cdot 3.$$

Ex. 9 The electrical resistance of a wire of given material varies directly as the length and inversely as the area of the cross section of the wire

Find the ratio of the electrical resistance of a wire 50 metres long and weighing 75 grams to that of a wire, of the same material, 100 ft long and weighing one ounce

1 metre $= 39 \cdot 37$ inches, and 1 kilog $= 2 \cdot 2$ lbs.

Let l denote the length, d thickness of the wire.

Electrical resistance $\propto \frac{l}{d^2}$, *i.e* $\frac{l}{r^2}$.

Weight $(w) = \rho\pi r^2 l$

Electrical resistance of wire

$$= m\frac{l}{r^2} = m\frac{l}{\frac{w}{\rho\pi l}} = \frac{ml^2\pi\rho}{w},$$

where m is a constant.

Electrical resistance of first wire

$$= \frac{m\rho\pi(50 \times 39 \cdot 37)^2}{\frac{75}{1000} \times 2 \cdot 2} = \frac{m\rho\pi(50 \times 3937)^2}{75 \times 22},$$

where weight and length are reduced to pounds and inches respectively.

Similarly, resistance of second wire

$$=\frac{m\rho\pi(100\times 12)^2}{\frac{1}{16}}=16m\rho\pi(100\times 12)^2$$

Required ratio

$$=\frac{m\rho\pi(50\times 3937)^2}{75\times 22}\div 16m\rho\pi(100\times 12)^2=1\cdot 0193$$

EXERCISES. X

1. In a certain fraction the difference between the numerator and denominator is 12, but if each be increased by 5 the value of the fraction becomes $\frac{3}{4}$. What is the fraction?

2 If a mixture of gold and silver in which 0·875 is gold, is worth £15. What will be the value of a mixture of equal weight in which 0·625 is gold? Assuming that the value of gold is 16 times that of silver.

3 If a fraction be such that its denominator exceeds twice its numerator by unity, prove that if its numerator and denominator be each increased by unity, the result will be $\frac{1}{2}$.

4 When unity is added both to the numerator and to the denominator of a certain fraction the result is $\frac{3}{2}$; but when unity is subtracted the result is 2. Find the fraction.

5. Divide £1015 among A, B and C, so that B shall receive £5 less than A, and C as many times B's share as there are shillings in A's share.

6. Two passengers have together 500 lbs of luggage and are charged 5*s* and 5*s*. 10*d* respectively for the excess above the weight allowed. If the luggage had all belonged to one of them he would have been charged 15*s*. 10*d*. How much luggage is a passenger allowed free of charge?

7. A sum of £3000 is to be divided among A, B and C. If each had received £1000 more than he actually does, the sums received would be proportional to the numbers 4, 3, 2. Determine the actual shares.

8. Divide 279 into two parts; such that one-third of the first part is less by 15 than one-fifth of the second part.

9. A person lends £5000 at a certain rate of interest. At the end of one year the principal is repaid together with the interest He then spends £25, and lends the remainder at the same rate of interest as before. At the end of one year more the principal and interest amount to £5382; find the rate of interest.

10. A sum of money amounts to £546 in three years at simple interest, and to £746 in seven years. Find the sum and rate per cent.

11. A body is made up partly of brass and partly of iron; if the brazen parts had been iron, and the iron parts brass, its weight would have been $\frac{21}{19}$ths of what it actually is. Given that the weights of equal volumes of brass and iron are as 9 to 7, find how much of the volume is of iron, and how much of brass.

12. Divide the number 500 into two parts such that the sum of $\frac{1}{5}$th the greater and $\frac{1}{7}$th the smaller shall be less than the difference of the parts by 60

13 The volumes of two right cylinders are as 11 . 8, the height of the first is to that of the second as $3:4$. If the base of the first has an area 16·5 sq. ft, what is the area of the base of the second?

14. Between one census and the next, the native population of a town increased by 8 per cent, while the foreigners decreased from 200 to 150. The increase in the total population was 7 per cent.; what was the total population of the second census?

Quadratic equations.—As already indicated (p. 68), when a given equation expressed in its simplest form involves the *square* of the unknown quantity it is called a **quadratic** equation Such an equation may contain only the square of the unknown quantity, or it may include both the square and the first power

Ex. 1 Solve the equation $x^2-16=0$

$$\therefore x^2=16, \quad x=\pm 4.$$

It is necessary to insert the double sign before the value obtained for x, as both $+4$ and -4 when squared give 16

The solution of a given quadratic equation containing both x^2 and x can be effected by one of the three following methods.

First method.—The method most widely known, and generally used, may be stated as follows:

Bring all the terms containing x^2 and x to the left-hand side of the equation, and the remaining terms to the right-hand side.

Simplify, if necessary, and divide all through by the coefficient of x^2

Finally, add the square of one-half the coefficient of x to both sides of the equation, take the square root of both sides, and the required roots can be readily obtained.

Ex. 2. Solve the equation $x^2-11x-26=0$.

$$x^2-11x=26 \qquad \text{(i)}$$

Add to each side one-half the coefficient of x;

$$\therefore\ x^2-11x+\left(\frac{11}{2}\right)^2=26+\frac{121}{4}=\frac{225}{4},$$

or

$$\left(x-\frac{11}{2}\right)^2=\left(\frac{15}{2}\right)^2;$$

$$x-\frac{11}{2}=\pm\frac{15}{2}; \qquad \text{(ii)}$$

$$\therefore\ x=\frac{11}{2}\pm\frac{15}{2}=13 \text{ or } -2.$$

Second method.—What may be termed the second and the third methods of solution may be indicated in the following manner Where the given equation can be resolved into factors, then the value of x which makes either of these factors vanish, is a value of x which satisfies the given equation

Ex. 3. Solve the equation $x^2-11x-26=0$.

Since $x^2-11x-26=(x+2)(x-13)=0$,

$$x-13=0, \text{ when } x=13,$$

$$\text{and } x+2=0, \text{ when } x=-2.$$

Hence $x=13$ or $x=-2$ is a solution of the equation and 13 and -2 are the roots of the given equation

Third method.—A given equation can be written in the form $y=f(x)$, p 68 Substitute values for x and calculate corresponding values of y Plot on squared paper and draw a curve through the plotted points. Then **as a function can only change sign when x passes through one of its roots,** it follows that the points of intersection of the curve with the axis of x are the roots required

The general solution may be obtained as follows:—

$f(x)$ may be written $\quad ax^2+bx+c=0.$

Then $$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Adding to each side the square of half the coefficient of x, or $\left(\frac{b}{2a}\right)^2$, we have

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2};$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\qquad\text{(i)}$$

The following important cases occur.

If b^2 is greater than $4ac$, i.e. $b^2>4ac$, there are two values of x, or roots, satisfying the given equation and the curve cuts the axis in two points

If $b^2=4ac$ the two roots are equal and the curve touches the axis; each is $-\frac{b}{2a}$.

If $b^2<4ac$, there are no real values which satisfy the given equation, and the roots are said to be imaginary, and the curve does not meet the axis.

Ex 4 $2x^2-8x+6=0$.

Solving this equation in the usual manner, the roots of the equation are found to be 1 or 3.

Or, by substitution in the formula,

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a},$$

$a=2,\ b=-8,\ c=6$;

$$\therefore\ x=\frac{8}{4}\pm\frac{\sqrt{84-4\times2\times6}}{4}$$

$$=2\pm1=1\text{ or }3.$$

Ex. 5. $2x^2-4x+2=0$

$$x=\frac{4}{4}\pm\frac{\sqrt{16-4\times2\times2}}{4};$$

$$\therefore\ x=1.$$

In this equation $b^2=4ac$.

Ex. 6. $2x^2-4x+3=0$.

Here $a=2,\ b=-4,\ c=3$.

$$x=\frac{4}{4}\pm\frac{\sqrt{16-4\times2\times3}}{4}.$$

Here $b^2<4ac$, and the roots are imaginary.

All these results are readily understood by using squared paper

Let $y=f(x)$, then for a series of values of x the corresponding values of y can be calculated The curve passing through the plotted points will for all positive values of y lie above the axis of x and below for negative values. In passing from positive to negative values the curve must obviously cross, or intersect, the axis of x. Each such point gives a value of x which satisfies the given equation, or, is a root of the equation.

Thus, by making the graph of $y=f(x)$, and measuring the intercepts on the axis of x, we may obtain approximately the values of x which make y equal to zero.

By assuming values of x in the neighbourhood of such a point, or points, and plotting the values obtained for y to a larger scale, a solution of a given equation to any desired degree of accuracy can be obtained

The two points of intersection may coincide; the axis of x is then a tangent to the curve This corresponds to the case of equal roots.

The plotted curve may not touch, or intersect, the axis of x, the values or roots of the given equation are then said to be imaginary

Ex. 7. Solve the equation $x^2-4\,79x+4\,843=0$

Let $\qquad y=x^2-4{\cdot}79x+4{\cdot}843$

When $\qquad x=0,\ y=4\,843;$

when $\qquad x=1,\ y=1{\cdot}053.$

Substitute other values for x; calculate values of y and tabulate as follows.

x	0	1	2	3	4
y	4·843	1·053	-0·737	-0 527	1·683

From the tabulated values of x and y a change of sign is seen to occur in passing from $x=1$ to $x=2$, and again from $x=3$ to $x=4$. It is clear that one root lies between each pair of these values. Plot the tabulated values of x and y, the curve representing the equation passes through the plotted points and intersects the axis of x at points E and F (Fig. 16). By measuring the distances of these points from the origin we

obtain the values of x, or roots which satisfy the equation. These are found to be 1·45 and 3·34 respectively. If required to find the numerical values of the roots to a higher order of

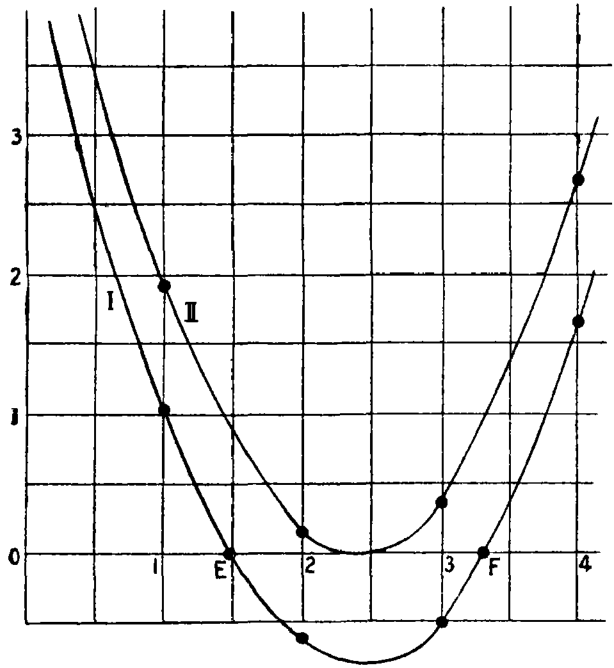

Fig. 16

accuracy than three figures, then the curve near to E and F may be plotted to a larger scale, and the values of x determined to any necessary degree of accuracy.

Ex. 8. Solve the equation $x^2 - 4{\cdot}79x + 5{\cdot}736025 = 0$

As before, values of y corresponding to various values of x should be calculated and tabulated as follows:

x	0	1	2	3	4
y	5·736	1·946	0·156	0·366	2·576

Plot these values and draw a curve passing through the plotted points. It touches the axis of x at the point $x=2\cdot395$ (approx.). It will be noticed that in this case $b^2=4ac$, as on p. 90.

If the value of c is increased, b and a remaining the same, then the roots are imaginary, and the curve does not cut the axis of x

Another graphical method may be used to obtain the solution of a quadratic equation.

Let the equation be,

$$x^2-bx+c=0. \quad \text{............ ....(i)}$$

Set off on squared paper from any convenient point O a distance $OA=b$; draw AB, equal to c, and OD, equal to unity, perpendicular to OA (Fig. 17).

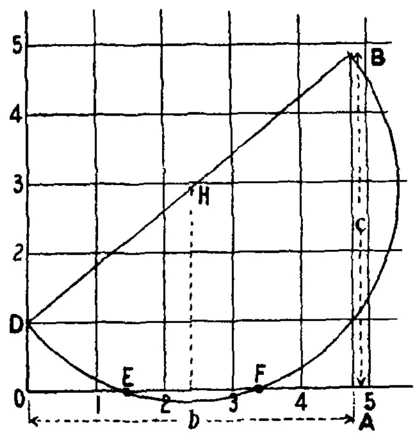

FIG 17

Join DB; and on DB as diameter describe a semicircle. The two points of intersection of the semicircle with the line OA are two roots required.

Ex. 9. Solve the equation, $x^2-4\cdot79x+4\cdot843=0$.

Comparing this equation with (i) it is seen that $b=4\cdot79$, $c=4\cdot843$. Hence, make $OA=4\cdot79''$ and $AB=4\cdot843.''$ Finally, $OD=1''$ Join BD. Then a semicircle described on BD as diameter cuts the line OA at points E and F, where $OE=1\cdot45''$ and $OF=3\cdot34''$, giving the two values required.

Ex. 10. Solve the equation $x^2 - 4{\cdot}79\,x + 5{\cdot}736 = 0$.

Setting off AB equal to $c = 5{\cdot}736$, the semicircle, on DB as diameter, touches OA approximately, or, in other words, the two points of intersection are coincident, and the quadratic has two equal roots. If c be increased, b remaining constant, the semicircle moves away from the line OA, and the roots become imaginary

A proof of the preceding construction may be obtained as follows:

Let D' denote the point of intersection of the semicircle with the vertical through B. Then because the centre of the semicircle bisects DB, $OE = FA$, and $AD' = OD = $ unity

By property of chords of a circle $AF \cdot AE = AD' \cdot AB$,

$$\therefore\ OE \,.\, EA = AB = c, \text{ but } OE + EA = b\,;$$

$$\therefore\ OE \text{ and } EA \text{ are the roots required.}$$

If c is negative AB must be drawn in the direction opposite to OD.

If c is positive, the roots *may be* either real or imaginary

If c is negative, the roots *must* be real.

Equations which may be solved as quadratics.—Much unnecessary labour will result if the attempt is made to obtain unity as the coefficient of x^2 in all equations. It may be found better to use another letter, such as y or z, and then to proceed to solve the equation in the ordinary manner, finally solving the equation for x. The following examples will illustrate some of the methods which may be adopted

Ex. 11. Solve

$$40\left(x + \frac{1}{x}\right)^2 - 286\left(x + \frac{1}{x}\right) + 493 = 0. \quad \ldots \quad \text{(i)}$$

Put
$$y = x + \frac{1}{x}. \quad \ldots \quad \text{(ii)}$$

The equation becomes

$$40y^2 - 286y = -493\,;$$

$$\therefore\ y^2 - \frac{143}{20}y = -\frac{493}{40}\,;$$

$$\therefore\ y^2 - \frac{143}{20}y + \left(\frac{143}{40}\right)^2 = -\frac{493}{40} + \left(\frac{143}{40}\right)^2 = \frac{729}{1600};$$

$$\therefore\ y = \frac{143}{40} \pm \frac{27}{40} = \frac{17}{4}, \text{ or } \frac{29}{10}$$

From (ii), $\frac{17}{4}=x+\frac{1}{x}$;

$$\text{or } x^2-\frac{17}{4}x=-1,$$

$$x=\frac{17}{8}\pm\frac{15}{8}=4, \text{ or } \frac{1}{4}.$$

Putting $x+\frac{1}{x}=\frac{29}{10}$,

then $x=2\frac{1}{2}$, or $\frac{2}{5}$.

Hence the values are 4, $\frac{1}{4}$, $2\frac{1}{2}$, or $\frac{2}{5}$.

Ex. 12. Solve the equation $(x^2-4x+3)^2-8(x^2-4x+3)=0$.

Writing y for x^2-4x+3, the given equation becomes

$$y^2-8y=0;$$

$$\therefore y^2-8y+(4)^2=16.$$

$$y=4\pm4=8, \text{ or } 0.$$

Substitute these values for y, then

$$x^2-4x+3=8,$$

$$x=2\pm3=5, \text{ or } -1.$$

Similarly, the second value for y gives

$$x^2-4x+3=0;$$

$$\therefore x=2\pm1$$

$$=3, \text{ or } 1.$$

The values of x are 1, 3, 5, -1.

Instead of using the letter y the equation could be solved directly, thus

$$(x^2-4x+3)^2-8(x^2-4x+3)=0;$$

$$x^2-4x+3-8=0, \quad \ldots\ldots \quad \ldots\text{(i)}$$

$$\text{or } x^2-4x+3=0. \quad \ldots\ldots \quad \ldots\text{(ii)}$$

From (i), $x^2-4x-5=0$;

$$\therefore (x-5)(x+1)=0.$$

Hence, $x=5$, or -1.

From (ii), $x^2-4x+3=0$,

$$\text{or } (x-1)(x-3)=0;$$

$$x=1, \text{ or } 3.$$

Ex. 13. Solve $x^2+\frac{9}{x^2}-4\left(x+\frac{3}{x}\right)-6=0$.

By adding 6 to each side of the equation, the quantity on the left of the brackets becomes the square of the quantity enclosed by the brackets.

Thus, $$x^2+\frac{9}{x^2}+6-4\left(x+\frac{3}{x}\right)=6+6=12. \quad \quad \text{(i)}$$

Let $$y=x+\frac{3}{x}. \quad \quad \text{(ii)}$$

Then (i) may be written
$$y^2-4y=12,$$
$$\text{or} \quad y^2-4y+(2)^2=12+4=6;$$
$$y=2\pm4=6, \text{ or } -2$$

Substitute these values in (ii) Thus, when $y=6$,
$$x+\frac{3}{x}=6,$$
$$\text{or} \quad x^2-6x+3=0;$$
$$x=3\pm\sqrt{6}.$$

Again, $$x+\frac{3}{x}=-2,$$
$$x^2+2x+(1)^2=-3+1=-2,$$
and the roots are imaginary.

The values satisfying the given equation are
$$x=3\pm\sqrt{6}=5\ 45,\ 0{\cdot}55.$$

Equations reducible to quadratics.—Equations of the fourth degree can in some cases be solved as two quadratic equations.

Ex. 14. Solve $x^4-17x^2+16=0$.

The equation may be written
$$(x^4-8x^2+16)-9x^2=0, \text{ or } (x^2-4)^2-(3x)^2=0;$$
$$\therefore\ (x^2+3x-4)(x^2-3x-4)=0.$$

Hence, $$x^2+3x-4=0, \quad \quad \text{(i)}$$
or $$x^2-3x-4=0. \quad \quad \text{(ii)}$$

From (i), $$x^2+3x-4=(x+4)(x-1);$$
$$x=-4, \text{ or } 1$$

From (ii), $$x^2-3x-4=(x-4)(x+1);$$
$$\therefore\ x=4, \text{ or } -1.$$

The values of x which satisfy the given equation are
$$x=\pm4,\ x=\pm1.$$

Relations between the coefficients and the roots of a quadratic equation.—In the preceding examples we have been able, from a given quadratic equation, to find the roots, or the values, which satisfy the given equation The converse of this is often required, *i e* to form a quadratic equation with given roots

It has been already seen that if we can resolve the left-hand side of the given equation, when reduced to its simplest form, into factors, then the value of x which makes either of these factors zero, is a value of x which satisfies the given equation

Thus, the roots of the equation $(x-\alpha)(x-\beta)=0$ are α and β

Conversely, an equation having for its roots α and β is

$$(x-\alpha)(x-\beta)=0.$$

Hence, if α and β denote the roots of the equation,

$$ax^2+bx+c=0$$

We have $$ax^2+bx+c=a(x-\alpha)(x-\beta),$$

$$ax^2+bx+c=a(x^2-\alpha x-\beta x+\alpha\beta)$$

$$=a\{x^2-(\alpha+\beta)x+\alpha\beta\}$$

Comparing coefficients on both sides

$$a(\alpha+\beta)=-b \text{ and } a\alpha\beta=c,$$

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a},$$

therefore, when the coefficient of x^2 is unity, the sum of the roots is equal to the coefficient of x, and the product of the roots is equal to the remaining term.

Ex 15 Form the quadratic equations having roots 1 and 4.

Here $$(x-1)(x-4)=x^2-5x+4;$$

$\therefore$ Required equation is $x^2-5x+4=0$.

Ex 16 Form the quadratic equation having roots

$$-3+\sqrt{2} \text{ and } -3-\sqrt{2}$$

Here we have $(x+3-\sqrt{2})(x+3+\sqrt{2})=(x+3)^2-2$;

$\therefore$ the required equation is $x^2+6x+7=0$.

Ex. 17. Form the quadratic equation having roots a and $\frac{1}{a}$.

Here $$(x-a)\left(x-\frac{1}{a}\right);$$

required equation is $x^2-\frac{a^2+1}{a}x+1=0$.

EXERCISES. XI.

Solve the equations:

1. $x^2-5x+4=0$
2. $x^2-6x+8=0$
3. $x^2+7x+12=0$
4. $x^2-7\cdot08x+11\cdot875=0$
5. $x^2-6\cdot09x+9\cdot179=0$
6. $\frac{x}{2}+\frac{x-4}{x+4}=\frac{x}{3}$.
7. $\frac{5}{x}+\frac{x-7}{x^2}=\frac{11}{9}$
8. $\frac{3x^2-27}{x^2+3}+\frac{90+4x^2}{x^2+9}=7$.
9. $x^2+6x-35=0$.
10. $\frac{9}{x}+\frac{25x}{x-1}+9=0$
11. $m\left(x-\frac{1}{x}\right)+n\left(x+\frac{1}{x}\right)=0$
12. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{a-x}+\frac{1}{b-x}$

13 Prove that the roots of $x^2+px+q=0$ are equal when $p^2-4q=0$, also that one is half the other, if $9q=2p^2$

14. If α and β are the roots of the equation $x^2+px+q=0$, express $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$ in terms of p and q

15 Solve the quadratic equation
$$x=\frac{16}{15}+\frac{1}{x}$$

Solve the equations:

16 $x^2+\frac{1}{x^2}+\frac{1}{3}\left(x+\frac{1}{x}\right)=3\frac{5}{12}$.

17. $x^2+y^2+4x-6y-13=0$,
$3x-2y-1=0$

18 Find the roots of the equation $x^2+7x\sqrt{2}=60$, first in a surd form and then in a decimal form.

19. Form the quadratic equation whose roots are $3+\sqrt{2}$ and $3-\sqrt{2}$.

Solve the equations:

20 $x+a=\sqrt{\{a^2+x\sqrt{(2x^2-a^2)}\}}$

21 $2x^2-3x-\sqrt{(4x^2-6x-1)}=2$.

22 $x^4-4x^2+3=0$.

23. $x+\dfrac{4a}{x+1}=2a+1.$ **24.** $x^2-5x+6=24-2\sqrt{(x^2-5x+6)}.$

25. $x+\dfrac{1}{x}=2(1+\sqrt{2}).$ **26.** $x^2+4x+\sqrt{(x^2+4x+10)}=2.$

27 $\dfrac{2x-3}{5}+\dfrac{1}{7}\left(5x-\dfrac{6x+4}{5x+1}\right)=x+\dfrac{5x+8}{3x-14}+\dfrac{1}{3}\left(\dfrac{x}{7}+\dfrac{x-9}{5}\right).$

28. $x^2+2\sqrt{x^2+2x+3}=12-2x.$

29. $cx+\dfrac{ac}{a+b}=(a+b)x^2.$ **30.** $2{\cdot}3x^2-6{\cdot}72x-13{\cdot}6=0$

31 $0{\cdot}24x^2-4{\cdot}37x-8{\cdot}97=0$

32 $zx=y^2,\ x+y+z=21,\ x^2+y^2+z^2=189$

33. $x^3-2x^2-3x+4=0$ **34** $x^2-5{\cdot}17x+5{\cdot}985=0.$

35 $x=1+\cfrac{1}{2+\cfrac{1}{1+x}}.$ **36** $\dfrac{1}{1+x}+\dfrac{1}{2+x}=\dfrac{1}{1-x}+\dfrac{1}{2-x}.$

37 $x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}=4.$

38 Find to three places of decimals, by the use of squared paper, the roots of the equation $x^2-5{\cdot}45x+7{\cdot}181=0$

39 $x^2-2x\sqrt{3}+2=0.$

40 Show that if A and B are the roots of the equation $x^2-px+q=0$, then will $p=A+B$ and $q=AB$. Form the equation whose roots are 27 and -13

41 Prove that the equation

$$\frac{(x^2-x+1)^3}{x^2(x-1)^2}=\frac{(a^2-a+1)^3}{a^2(a-1)^2}$$

is satisfied by

$$x=a,\ \frac{1}{a},\ \frac{a-1}{a},\ \frac{a}{a-1},\ \frac{1}{1-a},\ 1-a$$

42 Prove that the roots of the equation

$$x^4-4x^2+1=0$$

are

$$\pm\sqrt{\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)}\ \text{ and }\ \pm\sqrt{\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)}$$

Simplify these roots to a form suitable for numerical computation, and calculate each of them to three decimal places.

Simultaneous Quadratics.—Equations involving the squares of two unknown, or variable, quantities, such as x^2 and y^2, may be solved by methods similar in many respects to those adopted in the case of equations of the first degree. That is to say, we can, by multiplication, division, or substitution, obtain an equation involving only one unknown quantity. From this equation the value of the unknown quantity can be determined, and by substitution the value of the remaining unknown can be found

If a given equation contains a factor of the form $x+y$, we may proceed to obtain $x-y$, and finally the separate values of x and y may be obtained by addition or subtraction

Ex 1 $\qquad x+y=11, \qquad$ (i)

$\qquad xy=30. \qquad$ (ii)

From (i), $\qquad x^2+2xy+y^2=121 \qquad$ (iii)

Multiply (ii) by 4, $\qquad 4xy \quad =120 \qquad$ (iv)

Subtract (iv) from (iii), $\qquad x^2-2xy+y^2= \quad 1,$

$\qquad$ or $x-y=\pm 1.$

Hence, $\qquad x+y=11,$

$\qquad x-y=\pm 1;$

With the upper sign

$$\left.\begin{aligned} x+y&=11,\\ x-y&=\ \ 1,\end{aligned}\right\}\text{gives } x=6 \quad y=5,$$

With the lower sign

$$\left.\begin{aligned} x+y&=\ \ 11,\\ x-y&=-1,\end{aligned}\right\}\text{gives } x=5, \ y=6$$

Ex 2. Solve the equations

$\qquad x^2+xy=84, \qquad$ (i)

$\qquad xy+y^2=60. \qquad$ (ii)

Adding (i) and (ii),

$$x^2+2xy+y^2=144;$$

$$x+y=\pm 12.$$

Also from (i), $\qquad x(x+y)=84.$

and from (ii), $\qquad y(x+y)=60;$

Substitute, $\qquad \therefore \pm 12x=84$ and $\pm 12y=60;$

$$x=\pm 7 \text{ and } y=\pm 5.$$

The values required are $x=7$, $y=5$, $x=-7$, $y=-5$.

Ex. 3. Solve $$x^3+4xy+y^3=38, \qquad \text{(i)}$$
$$x+y=2. \qquad \text{(ii)}$$

From (ii), $y=2-x$. Substitute in (i);

$$\therefore\ x^3+4x(2-x)+(2-x)^3=38,$$
$$\text{or } 2x^2-4x-30=0;$$
$$x^2-2x-15=0,$$
$$\text{or } (x-5)(x+3)=0$$

Hence $x=5$, or -3; and from (ii), $y=-3$, or 5.

Or, we may proceed as follows

The given equation is $$x^3+y^3+4xy=38,$$
$$(x+y)(x^2-xy+y^2)+4xy=38.$$

But from (ii), $$x+y=2;$$

by division $$x^2-xy+y^2+2xy=19,$$
$$\text{or } (x+y)^2-xy=19,$$
$$xy=-15.$$

And thus, from (ii), $$x-y=8;$$
$$x=5,\ y=-3,\ x=-3;\ y=5$$

One pair of roots will be $x=2-y=\infty$.

It will be noticed that this solution gives a method by which the order of one equation may sometimes be reduced by using the other

EXERCISES XII.

1. $x^2-2x+y^2-2y=14,\quad xy=5.$

2. $x^2-4y^2=8,\quad 2(x+y)=7$

3. $3x^2+5xy-7x-3y=128,\quad 3y-2x=2.$

4. $3x+y=15,\quad 2x^2-3y^2=5.$

5. $\left.\begin{array}{r} x^2+xy-6y^2=6, \\ x^2+5xy+6y^2=30 \end{array}\right\}$

6. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{x+y}{12}=\dfrac{7}{x+y+5}$

7. $\left.\begin{array}{r} x^2-xy-y^2=\dfrac{xy}{15}, \\ x-y=2. \end{array}\right\}$

8. $4(x^2-y^2)=35,\quad x-2y=2$

9. $x^2+y=51,\quad 2x^2+y^2=102.$

10. $\left.\begin{array}{r} \sqrt{(x+y)}+\sqrt{(x-y)}=5, \\ \sqrt{(x^2-y^2)}=4.5 \end{array}\right\}$

11. $\left.\begin{array}{r} x^2+y=8, \\ 3x+2y=7. \end{array}\right\}$

12. $\left.\begin{aligned} x^{-1}+y^{-1}+z^{-1}&=13,\\ y^{-1}-x^{-1}&=1,\\ x^{-1}y^{-1}-2z^{-1}&=0.\end{aligned}\right\}$

13. $\left.\begin{aligned} x^3+y^3+z^3+3a^3&=0,\\ x+y+z&=0,\\ x^2y+b^3&=0.\end{aligned}\right\}$

14. $\left.\begin{aligned} x+y+z=yz&=12,\\ x^2=y^2+z^2\end{aligned}\right\}$

15. $\left.\begin{aligned} xy+x+y&=7,\\ xz+x+z&=8,\\ yz+y+z&=17.\end{aligned}\right\}$

16 $\left.\begin{aligned} x^2y+xy^2&=0\ 18,\\ x^3+y^3&=0\ 189.\end{aligned}\right\}$

17. $2y^2=2x^2+1=xy+2$

Problems leading to quadratic equations.—One of the greatest difficulties experienced by a beginner in Algebra is to express the conditions of a given problem by means of algebraic symbols. The equations themselves may be obtained more or less readily, since the conditions are generally similar to those already explained, but some difficulty may be experienced in the interpretation of the results derived from quadratic equations. Since a quadratic equation which involves one unknown quantity has two solutions, and simultaneous quadratics involving two unknown quantities may have four solutions, it is clear that ambiguity may arise It will be found, however, that although the equations may have four solutions, only one solution is as a rule applicable to the particular problem. The fact that several solutions can be found and only one applies to the problem is due to the circumstance that algebraic language is far more general than ordinary methods of expression Usually no difficulty will be experienced in deciding which of the solutions is applicable to the problem in hand

Ex. 1. A person bought a number of articles for £80, if he had received four more for the same price, they would have cost him £1 each less than he paid What number did he buy?

Let x denote the given number.

Then the price of each is $\frac{80}{x}$

If four more could be obtained for the same price, the price of each would be $\frac{80}{x+4}$

That is $$\frac{80}{x+4}=\frac{80}{x}-1.$$

Multiplying both sides of the equation by $x(x+4)$,

$$80x = 80(x+4) - x^2 - 4x;$$

$$\therefore x^2 + 4x = 320,$$

$$x^2 + 4x + 2^2 = 320 + 4 = 324;$$

$$x = -2 \pm 18 = 16, \text{ or } -20.$$

It is obvious that 16 is the number required.

The value -20 does not correspond with the conditions of the problem, and is therefore not admissible.

Ex. 2. An arrow is projected vertically upwards with a velocity of 96 feet per second. After what time is it at a distance of 80 feet above the ground?

The relation between initial velocity (V), space described (S), and time (t) is given by the equation

$$S = Vt - \frac{1}{2}gt^2$$

Take $g = 32$ and substitute the given values:

$$80 = 96t - \frac{1}{2} \times 32 \times t^2;$$

$$16t^2 - 96t = -80,$$

or $$t^2 - 6t + 3^2 = -5 + 9 = 4,$$

$$t = 3 \pm 2 = 5, \text{ or } 1$$

Both values are admissible; the value one second indicating that the arrow is at the height of 80 feet at the end of the first second. It continues to rise until it reaches its greatest height and then begins to descend, and is at a height of 80 feet above the ground at the end of 5 seconds.

Ex. 3. Find two numbers whose difference is 8 and product 240.

Let x denote the least number, then $x+8$ is the greater.

Then $$x(x+8) = 240,$$

or $$x^2 + 8x = 240.$$

Hence, $$x^2 + 8x + (4)^2 = 240 + 16 = 256,$$

$$x = -4 \pm 16 = 12,$$

and $x+8=20$, the greater number.

The rejected solution is $x = -20$, the greater number being $x+8 = -12$.

Ex. 4. If in the equation $ax^2+bx+c=0$, the relations between a, b and c are such that $a+b+3=0$, and $2a-c+1=0$, what must be the value of a in order that one of the roots may be 5, and what is then the value of the other root?

In the given equation $ax^2+bx+c=0$

On substituting the given values,

$$25a+5b+c=0, \qquad \text{(i)}$$
$$a+b+3=0, \qquad \text{(ii)}$$
$$2a-c+1=0. \qquad \text{(iii)}$$

Multiply (ii) by 5 and subtract from (i),

we obtain $$20a+c-15=0, \qquad \text{(iv)}$$
$$2a-c+1=0$$

Add (iii) and (iv), $$22a-14=0,$$
$$a=\frac{7}{11}.$$

And by substitution, $$c=\frac{25}{11},\ b=-\frac{40}{11};$$

$$\frac{7}{11}x^2-\frac{40}{11}x+\frac{25}{11}=0$$

This is the form of the equation corresponding to the conditions of the problem; $\therefore\ 7x^2-40x+25=0$,

or $$(7x-5)(x-5)=0,$$
$$\therefore\ x=5, \text{ or } \frac{5}{7}.$$

Ex 5. If $z=ax-by^3x^{\frac{1}{2}}$

If $z=1{\cdot}32$ when $x=1$ and $y=2$,

and if $z=8{\cdot}58$ when $x=4$ and $y=1$,

find a and b Then find z when $x=2$ and $y=0$.

Substitute the given values

$$1{\cdot}32=a-8b. \qquad \text{(i)}$$
$$8{\cdot}58=4a-2b. \qquad \text{(ii)}$$

Multiply (i) by 4 and subtract from (ii),

$$3{\cdot}3=30b,$$
$$b=0{\cdot}11, \text{ and from (i) } a=2{\cdot}2$$

Or, use the positive sign,

$$8{\cdot}58=4a+2b$$
$$5{\cdot}28=4a+32b$$
$$3{\cdot}3=-30b; \qquad b=-0{\cdot}11,\ a=2{\cdot}2.$$

Hence, the given relation becomes

$$z=2{\cdot}2x\mp 0{\cdot}11y^3x^{\frac{1}{2}}.$$

When $x=2$, $y=0$, then $z=2{\cdot}2\times 2=4{\cdot}4$.

In forming a system of algebraic equations of second degree from given data, it is, as in simple equations, a matter of little importance in many cases whether the given conditions are expressed in terms of one or more variables, but, in general, it is better to employ as few as possible.

Ex 6. Find a proper fraction such that twice the denominator exceeds the square of the numerator by 2, and the product of the sum and difference of the numerator and denominator is 325

Let $\frac{x}{y}$ denote the given fraction, then

$$x^2 = 2y - 2 \qquad \text{(i)}$$

Also,

$$(y + x)(y - x) = 325, \qquad \text{(ii)}$$

or

$$y^2 - x^2 = 325. \qquad \text{(iii)}$$

Add (iii) to (i);

$$\therefore\ y^2 = 2y + 323,$$

or

$$y^2 - 2y - 323 = 0,$$

$$(y - 19)(y + 17) = 0,$$

$$y = 19, \text{ or } y = -17$$

Substitute these values for y in (i),

when $y = 19$,

$$x^2 = 38 - 2 = 36;$$

$$x = \pm 6.$$

when $y = -17$,

$$x^2 = -34 - 2,$$

$$x = \pm\sqrt{-36}.$$

The latter value is clearly not admissible.

Hence, the fraction is $\frac{6}{19}$.

Ex. 7. There are two positive numbers whose sum is 6, and the ratio of the first to the second exceeds the ratio of the second to the first by 2; find the numbers

Let x denote one number and y the other Then the first condition that the sum of the two numbers is 6 gives the relation

$$x + y = 6. \qquad \text{(i)}$$

Also

$$\frac{x}{y} - \frac{y}{x} = 2,$$

or

$$x^2 - y^2 = 2xy. \qquad \text{(ii)}$$

Squaring both sides of (i),

$$x^2 + 2xy + y^2 = 36. \qquad \text{(iii)}$$

Adding (iii) *to* (ii), $2x^2=36$,

$$x=\pm 3\sqrt{2};$$

and from (i), $y=6\pm 3\sqrt{2}$.

The value of x $-3\sqrt{2}$ is inadmissible, since both numbers are positive Hence, the two numbers are $3\sqrt{2}$ and $6-3\sqrt{2}$.

Ex. 8. A person lends £1500 in two separate sums, at the same rate of interest. The first sum is repaid, with interest, at the end of eight months, and amounts to £936; the second sum is repaid, with interest, at the end of 10 months and amounts to £630. Find the separate sums lent and the rate of interest.

Let x and y denote the two sums lent, and r denote the rate per £ per annum:

$$x+y=1500, \qquad \text{(i)}$$

$$x+\frac{2}{3}rx=936, \qquad \text{(ii)}$$

and

$$y+\frac{5}{6}ry=630, \qquad \text{(iii)}$$

From (ii), $x(3+2r)=2808$, $x=\dfrac{2808}{3+2r}$

From (iii), $y(6+5r)=3780$; . $y=\dfrac{3780}{6+5r}$

Substituting in (i), $\dfrac{2808}{3+2r}+\dfrac{3780}{6+5r}=1500$,

or $1250r^2+1575r-99=0$; $(50r-3)(25r+33)=0$

The only admissible value is $r=\dfrac{3}{50}=\dfrac{6}{100}$

This gives, $x=£900$, $y=£600$

Ex 9 Twice the area of the square on the diagonal of a rectangle equals five times the area of the rectangle; find the ratio of the sides.

Let x and y denote the two sides of the rectangle.

Area of rectangle $=xy$.

Twice the area of the square on the diagonal is $2(x^2+y^2)$.

Then $5xy=2(x^2+y^2)$;

$$2x^2+2y^2-5xy=0,$$

or

$$x^2+y^2-\frac{5}{2}xy=0;$$

$$\therefore \left(x-\frac{1}{2}y\right)\left(x-2y\right)=0.$$

Hence, $x:y=1:2$ or $2:1$.

Hence, the sides are as $2:1$

Ex 10. When two equal rectangles are placed side by side it is found that the diagonal of the rectangle thus formed is three halves of the diagonal of one of the given rectangles. Find the ratio of the sides of one of the given rectangles.

Let the two rectangles be placed so as to form one rectangle $ABGF$ (Fig. 18)

Let the side $BC=x$ and the side $BA=y$

$$BF=\sqrt{(2x)^2+y^2},\ BD=\sqrt{(x^2+y^2)}$$

But $\quad BF=\frac{3}{2}BD$;

$$\sqrt{4x^2+y^2}=\frac{3}{2}\sqrt{x^2+y^2}$$

Squaring,

$$4x^2+y^2=\frac{9}{4}(x^2+y^2),$$

or $\quad 16x^2+4y^2=9x^2+9y^2$;

$$7x^2=5y^2$$

$$\frac{x^2}{y^2}=\frac{5}{7};$$

$$\frac{x}{y}=\sqrt{\frac{5}{7}}.$$

Giving $\quad x : y=\sqrt{5} : \sqrt{7}$

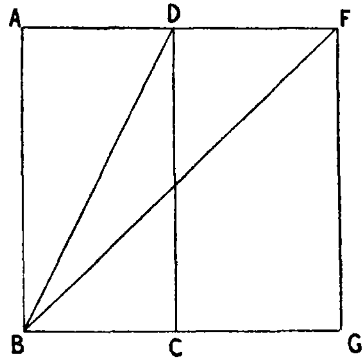

Fig. 18

EXERCISES XIII

1. Eight more articles can be obtained for £1 when the price is 5s. less per dozen Find the price

2. The area of a rectangle is equal to the area of a square whose side is three inches longer than one of the sides of the rectangle If the breadth of the rectangle be diminished by one inch and its length increased by two inches, the area is unaltered Find the lengths of the sides

3. The product of two numbers is 48 and the difference of their squares is to the sum of their cubes as 13 to 217. Find the numbers

4. The diagonal of a rectangular field is to its length as 13 to 12, and its area is 4860 square yards. Find its length and breadth

5 A certain sum of money had to be divided equally among 100 persons. If the sum had been increased by £5, each person would have received 5 per cent. more. What was the sum?

6. The area of a square, with the addition of 31 square feet, is equal to the area of a rectangle the sides of which are 2 and 3 feet respectively greater than the sides of the square. Find the length of a side of the square.

7. If a certain room were half as broad again as it is, it would be square; and if it were 3 ft. longer and 2 ft wider its area would be 6 square yards greater than it is. Find its length and breadth.

8. Find two numbers such that their product is 91, and the difference of their squares is to the difference of their cubes as 20 to 309

9 The area of a certain rectangle is equal to the area of a square whose side is 6 inches longer than the breadth of the rectangle The rectangle is such that if its breadth were decreased by 3 inches and its length increased by 9 inches, its area would be unaltered. Find the lengths of its sides

10 The sum of two numbers is 5, and the ratio of the square of the first to the square of the second is as 1 3 Find the numbers

11 Three numbers are as 1, 2, 3: the sum of their squares is 63 times the sum of the numbers Find them

Cubic equations.--When a given cubic equation can be resolved into its three factors, each of these factors will, when equated to zero, give a value of x which will satisfy the given equation. Each such value is therefore one of the roots required

Ex 1 Find the roots of the equation $x^3 - 3x^2 - 10x + 24$.

$$x^3 - 3x^2 - 10x + 24 = (x - 2)(x + 3)(x - 4).$$

Put each of the factors equal to zero, then

$$x - 2 = 0 ; \quad x = 2 ;$$

$$x + 3 = 0, \text{ or } x = -3 , \quad x - 4 = 0 , \quad x = 4.$$

Hence, the roots of the given equation are 2, - 3, 4

One method, which may often be used with a given cubic equation, is to bring all the terms of the equation to the left-hand side and simplify if necessary. Then, if by inspection, or by trial, one root can be obtained, the remaining roots may be obtained by solving the resulting quadratic equation

Ex. 2. Solve the equation $x^3 + 3x^2 - 6x = 8$

Bring all the terms to the left-hand side, and the equation becomes $x^3 + 3x^2 - 6x - 8 = 0$

By trial $x = 2$ satisfies the equation; hence, $x - 2$ is a factor. Dividing the given equation by $x - 2$, we obtain $x^2 + 5x + 4 = 0$, the factors of which are $(x + 1)(x + 4)$. Hence, the roots of the equation are $x = 2$, -1 and -4.

The methods just indicated become very laborious when the roots of an equation are not whole numbers; in such cases, as well as in those referred to, the values can be obtained by using squared paper.

Thus, Ex 1 may be written in the form

$$y = x^3 - 3x^2 - 10x + 24.$$

Put $x = 1$, 2, etc. The following values of y can be obtained

x	-3	-2	-1	0	1	2	3	4	5
y	0	24	30	24	12	0	-6	0	24

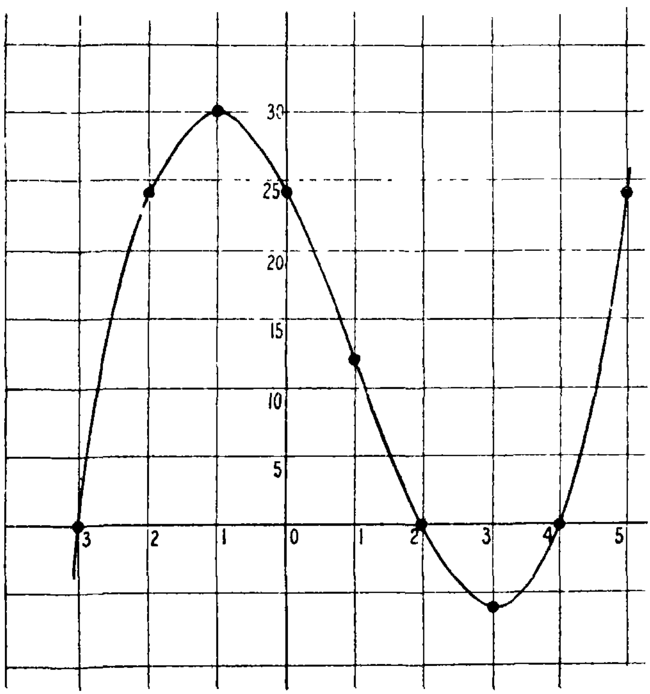

FIG. 19.—Graph of $x^3 - 3x^2 - 10x + 24 = 0$

Plot these values on squared paper and draw a fair curve through the plotted points as in Fig 19 Then the curve is

seen to intersect the axis of x at the three points $x=-3$, 2 and 4, and these are the roots required It should be noticed that on one side of each of these points the value of x gives a positive value for y and on the other a negative value; hence, we know that if for two assumed values of x the corresponding values of y are different in sign, then the root required lies somewhere between these values. If necessary, that portion of the curve lying between these assumed values may be plotted to a larger scale and the value of x obtained to any desired degree of accuracy.

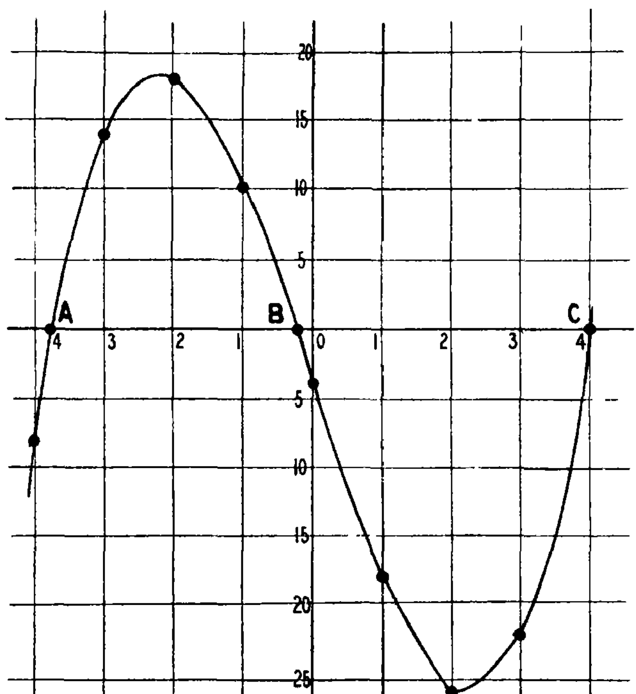

FIG 20 —Graph of $x^3-15x-4=0$

Ex. 3. Solve the equation $x^3-15x-4=0$.

Let $y=x^3-15x-4$. Substituting the values 0, 1, 2, . etc., for x, values of y can be calculated and tabulated as follows·

x	-4	-3	-2	-1	0	1	2	3	4	5
y	-8	14	18	10	-4	-18	-26	-22	0	46

Plot these values; then the curve passing through the plotted points (Fig. 20) is found to cross the axis of x between $x=-3$ and $x=-4$; also between 0 and -1, and at $x=4$. The values of x corresponding to $y=0$ can thus be obtained.

The roots of the given equation are found to be

$$x=-3.732, \quad -0.268, \quad 4$$

Ex 4 Solve the equation $x^3-0.25x-15=0$

We may write $\quad x^3-0.25x-15=y-y_1=0,$

where $\quad y=x^3, \qquad \text{(i)}$

and $\quad y_1=0.25x+15 \qquad \text{(ii)}$

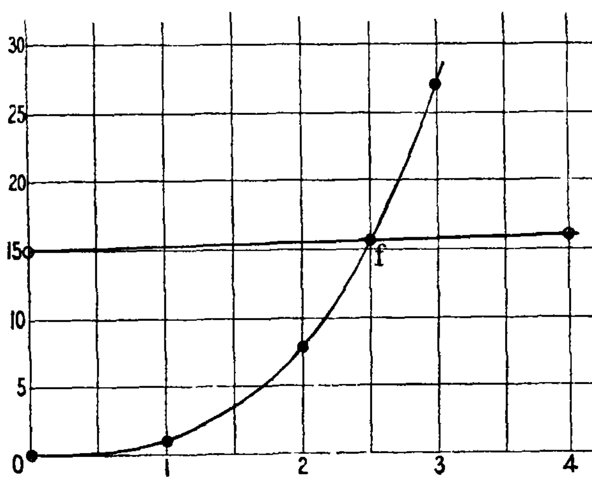

FIG 21 —Intersection of $y=x^3$ and $y_1=0.25x+15$

The solution is given when $y-y_1=0$ or the value of x determined by the point of intersection of the curve denoted by (i) and the line denoted by (ii) Thus, for values of x and corresponding values of y, the former will give a curve passing through the plotted points; the latter, a straight line The points of intersection of the line

and curve will give values of x which will satisfy the given equation.

Thus, from (i), when

$$x=0,\ y=0\ ;\ x=1,\ y=1\ ;\ x=2,\ y=8\ ;\ x=3,\ y=27.$$

From (ii), $x=0$, $y_1=15$; when $x=4$, $y_1=16$.

Plotting the former values we obtain the graph of $y=x^3$, the intersection of which with the graph of the latter values gives a point of intersection at f (Fig 21), where the value of $x=2{\cdot}5$, and this is one of the roots required. Other examples may be treated in like manner

If the given equation contains not only x^3 but also x^2, instead of a straight line we should have a second curve to be plotted; the intersection would give the value, or values, required

Ex 5 Find a value of x which satisfies the equation

$$x^2-5\log_{10}x-2{\cdot}531=0$$

As in the preceding cases, assuming values 1, 1·5, 2·0, 2·1 for x, we find the corresponding values of y change sign as x increases from 2·0 to 2·1. Hence, to obtain the value required, we may take x equal to 1·99, 2·00, 2·01 etc, and calculate values of y as in the following table:

x	1·99	2·00	2·01	2·02	2·03
y	−0·065	0·036	−0·007	0·023	0·052

Plot these values and draw a curve through the plotted points The curve is found to intersect the axis of x at a point $x=2{\cdot}012$. This is the value required

EXERCISES. XIV.

Solve the equations:

1. $x^3-12x^2-96x+512=0$

2 $x^3-2x^2-3x+4=0$. **3.** $8x^3-6x^2-3x+1=0$.

Find two roots of the equation·

4. $x^3-3a^2x+2a^3=0$.

Solve the equations:

5. $x^3-19x-30=0$. **6.** $x^3-15x-4=0$

7 $x^3-91x-330=0$. **8.** $x^3-12x^2+36x=7$

9. Find to two places of decimals the real positive root of
$$x^3+2x^2-4=0.$$

Find one root of each of the following equations:

10. $x^3+6x=20.$ **11.** $x^3-2x=5.$

12. $x^3-6x^2+18x=22.$ **13.** $x^3+9x-16=0.$

14. Show by plotting $y=x^4-4x^3-4x^2+16x+1$ between $x=-2$ and $x=4$ that the equation $x^4-4x^3-4x^2+16x+1=0$ has four real roots.

Find to two decimal places the value of the root which is numerically the greatest of the four.

15. Find two roots of the equation $x^3-12x=16$

16. Show by plotting that the equation $x^3-2{\cdot}4x^2-3x+7{\cdot}2=0$ has three real roots, and find the least positive value of x which satisfies the given equation.

17. Solve the equation $x^3-3x^2+2{\cdot}6=0.$

18. Find a value of x which satisfies the equation
$$x^2-5\log_{10}x-2{\cdot}531=0.$$

CHAPTER VII.

GRAPHS. SOME APPLICATIONS OF SQUARED PAPER

Graphs.—Any expression involving a variable, such as x, as well as known or unknown constants, may be briefly expressed by $f(x)$ [read as—function x]

Thus, we may write $f(x)=x^2-7x+12$.

The value of such a function may be denoted by y. Or $y=f(x)$, which is read as "y is a function of x."

Taking, for example, the former case

$$y=f(x)=x^2-7x+12,$$

then, by substituting various values for x, the corresponding values of y can be calculated The various values of y thus depend on those given to x, and x is called the **independent variable** and y the **dependent variable**

The line, straight or curved, which passes through the plotted points is called the **graph** of the function

In many cases a few points are all that are necessary to enable such a curve to be drawn with sufficient accuracy In the case of a straight line, two points are sufficient. It may be assumed that the reader is already familiar, from his previous work, with the linear equation

$$y=a+bx, \quad \text{...... ....} \quad (1)$$

in which, when x has the value 0, $y=a$, and the line makes the intercept on the axis of y equal to a

If a is zero, the equation becomes $y=bx$, and denotes a line passing through the origin

Use of squared paper.—When two variable quantities are connected by a relation such as $y=f(x)$, then, for assumed values of one, corresponding values of the other *can be calcu-*

lated. Using a sheet of squared paper, two convenient lines at right angles are assumed as axes, the simultaneous values may be represented by dots, or small crosses, and finally a curve passing through the plotted points may be drawn free-hand or by means of a flexible strip of metal or wood In a similar manner, a series of *experimental results* may be plotted and a curve drawn so as to pass as evenly as possible among the points. In other words, about an equal number of the results should lie on each side of any small portion of the curve. Such a curve may be assumed to give the most trust-worthy average for the constants in a general formula, the amount of deviation of any observation from this curve may, in the majority of cases, be assumed to be due to errors of observation

In all cases, except the equation of the first degree, in which the curve connecting the plotted point becomes a straight line, it is difficult to obtain the relation, or law, connecting x and y

By means of various artifices—some of which may be seen from the following examples—it is possible by plotting the logarithms x and y, or their reciprocals, etc., instead of their numerical values, to replace the curve by a straight line. From such a line the best average values for the two constants a and b in the equation $y=a+bx$ can be obtained

Thus, if two variables x and y are connected by the relation $y=ax^n$, where a and n are known constants, then when a is known or assumed, the curves corresponding to various values of n can be drawn

Thus, the equation $\mathbf{y=ax^n}$ becomes, when $a=1$, $\mathbf{y=x^n}$. Giving various values 1, 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, etc., to the index n, then functions of the form $y=x^3$, $y=x^{\frac{1}{3}}$, etc., are obtained Assuming values 0, 1, 2. for x, corresponding values of y can be found. The curves can be plotted, and are shown in Fig. 22. It will be seen that the curves $y=x^3$, $y=x^{\frac{1}{3}}$, and the straight line $y=x$ all intersect at the same point (1, 1)

It will also be noticed that, as the value of n is increased, the curve approaches closer and closer to the axis of x. Diminishing the value of n produces a similar effect with regard to

the axis of y. This fact is of some importance in proceeding to plot tabulated values of x and y, and more particularly to obtain the law, or relation, between x and y. Thus, if on plotting given values, a curve somewhat of the nature of the curve marked $n=2$ (Fig. 22) is obtained, then it would at once suggest that the numerical magnitude of n be diminished. If, on doing so, a curve of the form $\left(n=\frac{1}{2}\right)$ results, the probable value of n would lie somewhere between the two assumed values

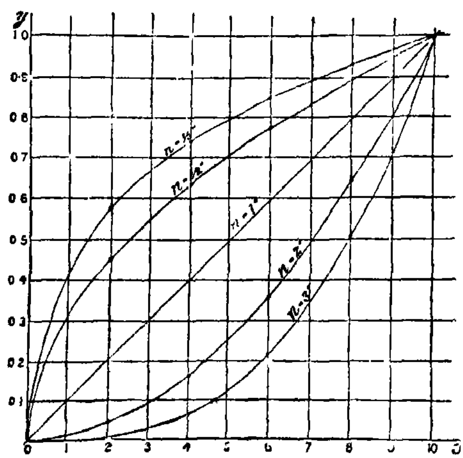

FIG 22.—Graph of $y=x^n$

In drawing a set, or family, of curves, as they are sometimes called, similar to the preceding examples, it will be found advantageous to use coloured pencils or crayons. Thus, the line "$n=1$" may be indicated in red, the curves below, where n is an integer, say in blue and green alternately; those above, where n is fractional, in green or yellow.

A good plan would be to draw on a piece of **transparent celluloid** a series of **standard curves** such as $y=x^n$ for various values of n, marking on each the value of n. This can be placed on a curve drawn through a series of plotted points, and the coincidence with one of the curves will suggest a probable value of n.

An important case of $y=ax^n$ .... (i)
occurs when n is negative. The equation then becomes

$$y=ax^{-n},$$

or

$$xy^n=a.$$

Assume a series of values for n, then for various values of x, corresponding values of y may be calculated, and the curves plotted

When $n=-1$, then (i) becomes

$$xy=a \quad \text{.........} \quad \text{(ii)}$$

For a definite numerical value for a the curve may be plotted.

The relation expressed by (ii) gives approximately the curve of expansion for a gas such as air at constant temperature, and is often taken to represent the curve of expansion of superheated or saturated steam

If p and v denote the *pressure* and *volume* respectively of a gas, instead of the form shown by (ii), the equation is usually written $pv=\text{constant}=c$, and is known as **Boyle's Law**; c is a constant, this is either given, or may be obtained from a pair of simultaneous values of p and v.

Ex 1. Plot the curve $xy=9$;

$$y=\frac{9}{x}. \quad \text{(ii)}$$

From (ii),

when $x=1$, $y=9$;

" $x=2$, $y=4.5$;

" $x=\frac{1}{1000}$, $y=9000$;

when x is very small, y is very great

Thus, let $x=\frac{1}{1000000}$,

then $y=9000000$

When $x=0$, then $y=\frac{9}{0}$, or is infinite in value. In other words, the curve gets nearer and nearer to the axis oy as the value of x is diminished, but does

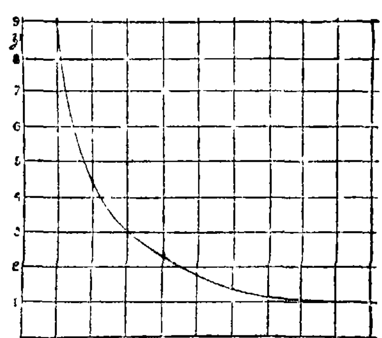

FIG. 23—Graph of $xy=9$

not reach the axis at any finite distance from the origin. This is expressed by the symbols $y=\infty$ when $x=0$.

As Eq. (ii) can be written $x=\frac{9}{y}$ it follows as before that when $y=0$, $x=\infty$.

The two lines, or axes, ox and oy are called **asymptotes**, and are said to meet (or touch) the curve at an infinite distance

Arranging in two columns a series of values of x and corresponding values of y obtained from Eq. (ii), we obtain,

Values of x,	0	1	2	3	4	5	6	7	8	9
Corresponding values of y,	∞	9	4 5	3	2 25	1 8	1 5	1 3	1 13	1

Plot these values of x and y on squared paper; the curve or graph passing through the plotted points is a **hyperbola**, as in Fig 23

One of the most important curves with which an engineer is concerned is given by the equation $pv^n=c$, where p denotes the pressure and v the volume of a given quantity of gas

The constant c and index n depend upon the substance used ; *i.e.* whether it is steam, air, etc

When, as in the preceding example, the values of c and n are known then for various values of one variable, corresponding values of the other can be obtained, and these can be plotted The plotted points will be found to lie on a straight line when n is 1 ; and on a curve when n is greater or less than 1 In the latter case, we may, by using logarithms, write the equation in the form

$$\log y=\log c+n \log x. \quad . \quad .. \qquad (\text{i})$$

This may be written $Y=C+nX$, or the equation to a straight line

Plot a series of values of $\log y$ and $\log x$ and join the points by a line ; then from two pairs of simultaneous values of Y and X the values of the constant c and n may be obtained.

It is not of course essential that the letters x and y should denote the two variables Other letters, such as p and v

(the initial letters of pressure and volume); Q and H; etc., may be used with advantage to suggest at once the quantities to which reference is made.

The converse problem may be stated. given various simultaneous values of p and v to calculate the numerical values of c and n.

To do this it is necessary to write the equation $pv^n = c$ in the form $\log p + n \log v = \log c$.

Plot $\log p$ and $\log v$ and draw a straight line lying evenly among the plotted points, and from two simultaneous values of p and v the values of c and n may be found

To take the case of the stuff in the cylinder of a steam or gas engine as an example. the pressure and volume are connected by an equation of the form pv^n = constant; from Tables the pressure corresponding to any given volume can be obtained, but unless the entries in such a table are very numerous it often happens that the volume corresponding to a given pressure, or the pressure corresponding to a given volume, cannot be found. The only means by which the required data can be arrived at is by a process of interpolation When values of p and v are plotted on squared paper and the curve lying among the plotted points is drawn, intermediate values can be at once obtained from the curve. The process of interpolation simply consists in reading from a given value of p, or v, the corresponding value of the remaining quantity

One objection to such a method is that errors may occur in plotting such a curve; another difficulty is experienced in reading the results with sufficient accuracy When the constants n and c in the general formula are found, values intermediate between those given by observation and, in some cases, even beyond them may be obtained by calculation. Some of the artifices which may be adopted to replace a curve by a straight line may be seen from the following examples.

Ex 2. The keeper of a restaurant finds when he has G guests in a day, his total daily expenditure (for rent, taxes, wages, wear and tear, food and drink) is E pounds and the total of his daily

receipts is R pounds. The following numbers are averages obtained by examination of his books on many days:

G	210	270	320	360
E	16 7	19 4	21·6	23 4
R	15 8	21 2	26 4	29·8

Find E and R and the day's profits, if he has 340 guests. What number of guests per day gives him just no profit? What simple algebraic law seems to connect E, R, P the profit, and G?

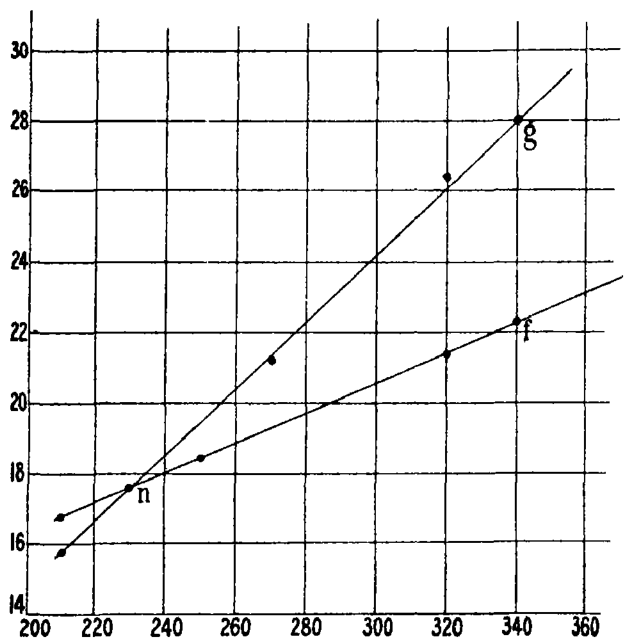

FIG 24

On plotting the given values of G and E, and G and R, it is seen that the curve joining the points is in each case a straight line, hence, the relation between E and G may be expressed by

$$G=a+bE, \qquad \text{(i)}$$

and between R and G by $G=c+dR$. ...... (ii)

Substitute in (i) the values at f and n (Fig. 24),

$$340 = a + 22.6b \quad . \quad . \quad . \text{(iii)}$$
$$230 = a + 17.3b$$

By subtraction,
$$110 = \quad 5.3b\,;$$
$$\therefore b = \frac{110}{5.3} = 20.75.$$

Substitute this for b in (iii) and obtain $a = -129$. Hence, the relation between E and G may be written,

$$G = 20.75E - 129.$$

Again, we may, in like manner, find the values of the constants c and d in Eq. (ii), by substituting the values at g and n;

$$\therefore 340 = c + 28 \quad d$$
$$230 = c + 17.3d$$
$$110 = \quad 10.7d\,;$$
$$d = \frac{110}{10.7} = 10.28$$

By substituting this value, we find $c = 52.2$.

Hence, the required relation is $G = 52.2 + 10.28R$

It will be obvious that the profit will be $R - E$. At the point n in the diagram R is equal to E; hence, 230 guests gives just no profit

In this manner we may find $P = 0.05G - 11.5$

Hence, the day's profits when the restaurant keeper has 340 guests is given by $P = 0.05 \times 340 - 11.5 = £5.5$

Ex. 3. Plot the curve $y = \frac{7.35x}{1 + 3.2x}$

Calculate the average value of y from $x = 0$ to $x = 8$

When $x = 2$, $y = \frac{7.35 \times 2}{1 + 3.2 \times 2} = \frac{14.7}{7.4} = 1.986$

When x is 0, 1, ..., values of y can be calculated and tabulated as follows

x	0	1	2	3	4	5	6	7	8
y	0	1.75	1.986	2.08	2.13	2.162	2.183	2.199	2.211

To obtain the average value we may use Simpson's Rule (p. 199). Thus,

sum of end ordinates 2.211,
" even " 8.191,
" odd " 6.299

Area from $x=0$ to $x=8$ is

$$\tfrac{1}{3}(2\cdot 211+8\cdot 191\times 4+6\cdot 299\times 2)=47\cdot 573\div 3=15\cdot 86.$$

But average value of y multiplied by length of base = area,

$$\text{average value of } y=\frac{47\cdot 58}{3\times 8}=1\cdot 982.$$

In some cases, when the expression $f(x)$ consists of several terms it may be advisable to arrange the various parts in a table and afterwards to add these to obtain the value of y. The method may be illustrated by a simple example as follows.

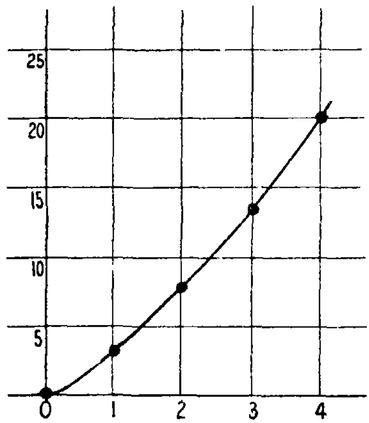

FIG. 25. Graph of $3x+\dfrac{x^2}{2}$.

Ex. 4. Draw the graph of the function $y=3x+\dfrac{x^2}{2}$.

The separate parts of the equation may be arranged in vertical columns. For various values of x the results should be obtained.

x	0	1	2	3	4
$3x$	0	3	6	9	12
$\dfrac{x^2}{2}$	0	0.5	2	4.5	8
y	0	3.5	8	13.5	20

and tabulated, and finally, by adding the numbers together in the vertical columns, the values of y are obtained. Plotting the tabulated values of x and y, a curve, as in Fig 25, is obtained.

Ex. 5. Experiments made to determine the (water) skin resistance of planks whose wetted surface is 100 square feet, yield the following results·

V = Speed per minute	200	400	600	800
R = Total resistance in lbs	3 28	11 7	24 6	41·7

Test whether the relation between R and V can be expressed by a law of the type $R \propto V^n$, and if so, find the values k and n in the formula $R = kSV^n$, in which S denotes wetted surface of plank Find the probable value of R when V is 1000

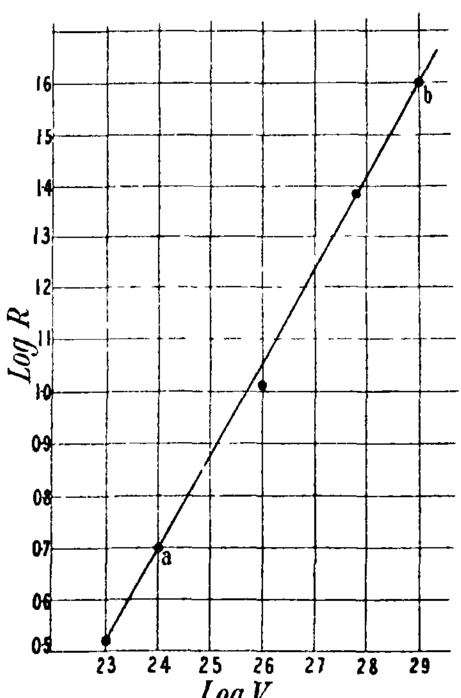

Fig. 26

Plot log V and log R on squared paper as in Fig 26, it will be found that a straight line can be drawn to lie evenly among the points, thus proving that the suggested formula is trustworthy. Now take a **strip of celluloid** on which a straight line is marked and draw a line such as ab, the intersection of the line with the axes will

determine the numerical values of the constants, or they may be obtained by calculation. The equation may be written:

$$\log R = \log k + \log S + n \log V.$$

At point a, $\log R$ is 0·7 and $\log V$ is 2·4, and at b the values are 1·6 and 2·9 respectively.

Hence, substituting these values, we have

$$1·6 = \log k + \log S + n \times 2·9 \quad .. \quad \text{(i)}$$

$$0·7 = \log k + \log S + n \times 2·4 \quad .. \quad \text{(ii)}$$

Subtracting, $\quad 0·9 = \qquad n \times 0·5,$

$$n = 1·8.$$

Substituting this value in (ii), as $\log S$ is 2·0, we get

$$0·7 = \log k + 2·0 + 1·8 \times 2·4,$$

$$\log k = 6·38, \quad \text{or} \quad k = 0·000002399$$

To find R when V is 1000, we have

$$\log R = \log k + \log S + n \log V;$$

$$\log R = 6·38 + 2·0 + 5·4 = 1·78,$$

$$R = 60·26 \text{ lbs}$$

Ex 6 The following numbers relate to the flow of water over a triangular notch

H	1·2	1·4	1·6	1·8	2·0	2·4
Q	4·2	6·1	8·5	11·5	14·9	23·5

H denotes the head of water (in feet), Q the quantity (in cubic feet) of water flowing per second Try if the relation between Q and H can be expressed in the form

$$Q = cH^n \quad \text{(i)}$$

If so, obtain the best average values of the constants c and n Also find Q when H is 2·2 and 2·6

The formula (i) may be written in the form

$$\log Q = \log c + n \log H \quad \text{(ii)}$$

Hence, if the relation given by (i) is true, on plotting (ii) a straight line will be obtained.

The given data may be arranged as follows:

H	1·2	1·4	1·6	1·8	2·0	2·4
Q	4·2	6·1	8·5	11·5	14·9	23·5
$\log H$	0·0792	0·1461	0·2041	0·2553	0·3010	0·3802
$\log Q$	0·6232	0·7853	0·9294	1·0607	1·1732	1·3711

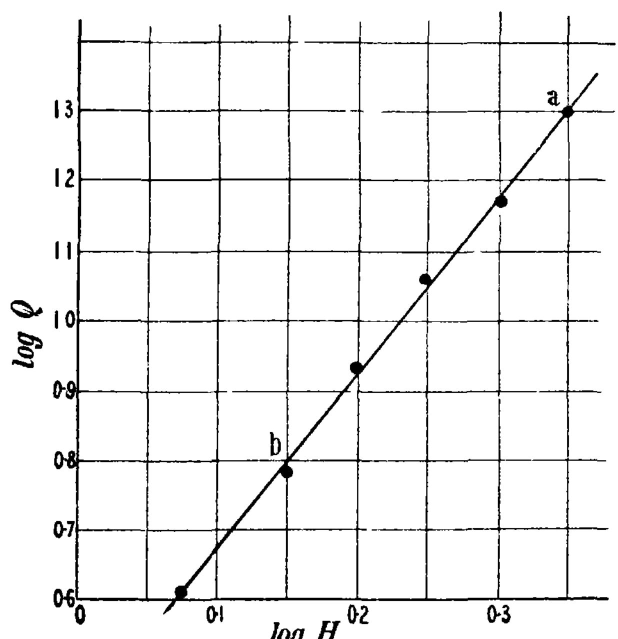

Fig. 27

Plot the last two rows as in Fig. 27, and a straight line may be drawn through the plotted points

By substituting in (ii) the values of $\log Q$ and $\log H$ from two

points such as a and b, the values of c and n may be obtained as follows:

$$\log Q = \log c + n \log H$$
$$1{\cdot}3 = \log c + n \times 0{\cdot}35$$
$$0{\cdot}8 = \log c + n \times 0{\cdot}15$$
$$0{\cdot}5 = 0{\cdot}2n,$$
$$n = \frac{5}{2}.$$

Substituting this value, we have

$$1{\cdot}3 = \log c + \frac{5}{2} \times 0{\cdot}35;$$
$$\log c = 0{\cdot}425, \text{ or } c = 2{\cdot}66$$

Hence, (1) may be written $Q = 2{\cdot}66\, H^{\frac{5}{2}}$

When H is 2·2, then we have

$$\log Q = \log 2{\cdot}66 + \frac{5}{2} \log 2{\cdot}2 = 1{\cdot}2809;$$
$$Q = 19{\cdot}09 \text{ cub. ft.}$$

Similarly, when H is 2·6, Q is found to be 29 cub. ft.

Ex. 7. In some experiments in towing a canal boat the following observations were made; P being the pull in pounds and v the speed of the boat in miles per hour. Find an approximate formula connecting P and v.

P	76	160	240	320	370
v	1·68	2·43	3·18	3·60	4·03
$\log P$	1·881	2·204	2·380	2·505	2·568
$\log v$	0·225	0·386	0·502	0·556	0·605

Plot $\log P$ and $\log v$ on squared paper and draw a line evenly through the plotted points. The equation to such a line may be written $\log P = n \log v + \log c.$

Substituting simultaneous values,

$$2{\cdot}568 = 0{\cdot}6\, n + c$$
$$1{\cdot}9 \quad = 0{\cdot}225\, n + c$$

Subtracting, $\qquad 0{\cdot}668 = 0{\cdot}375\, n,$

or $$n = \frac{668}{375} = 1{\cdot}78$$

Also, by substitution, $\log c = 2{\cdot}568 - 0{\cdot}6 \times 1{\cdot}78 = 1{\cdot}5 = \log 31{\cdot}6$

Hence, the formula required is $P = 31{\cdot}6\, v^{1{\cdot}78}$.

Ex. 8. **For the years 1896-1900, the following average numbers are taken from the accounts of the 34 most important electric companies of the United Kingdom**

U, means millions of units of electric energy sold to customers

C, means the total cost in millions of pence, and includes interest (7 per cent.) on capital, maintenance, rent, taxes, salaries, wages, coal, etc.

U	0·67	1·00	1·366	1·46	2·49
C	4·84	6·25	8·60	9·11	14·25

Is there any approximately correct simple law connecting U and C? If so, what is it? Assume that from the beginning there was the idea of, at some time, reaching a maximum output of 13·9, so that $U \div 13{\cdot}9$ is called f, a certain kind of *load factor*. Let $C \div U$ be called c the total cost per unit; is there any law connecting c and f?

Using the given values of U and C we may proceed to find the values of $f = U \div 13{\cdot}9$ and $C \div U = c$, and arrange as in the following table.

U	0·67	1·00	1·366	1·46	2·49
C	4·84	6·25	8·60	9·11	14·25
$f = U \div 13{\cdot}9$	0·048	0·072	0·098	0·105	0·18
$c = C \div U$	7·22	6·25	6·29	6·24	5·72
$\frac{1}{f}$	20·7	13·9	10·2	9·52	5·58

Plotting the given values of U and C, a straight line may be drawn among the plotted points. Its equation may be written $U = aC + b$. Substituting simultaneous values of U and C obtained from the curve, we find

$$1 = a \times 6{\cdot}4 + b$$
$$2 = a \times 12 + b$$

Subtracting, $\quad 1 = 5{\cdot}6a,$

$$a = 0{\cdot}18.$$

And, by substitution, $\quad b = -0{\cdot}16$

Hence, the simple approximate law connecting U and C may be written $U = 0{\cdot}18C - 0{\cdot}16$

In a similar manner plotting c and $\frac{1}{f}$, the relation $c = 5{\cdot}56 + \frac{0{\cdot}06}{f}$ is obtained.

Ex. 9. It is known that the relation connecting the pressure p and specific volume u of water-steam can be stated approximately as $pu^n = c$.

Test the accuracy of this rule for pressures ranging from 20 lbs. to 90 lbs. per sq. in

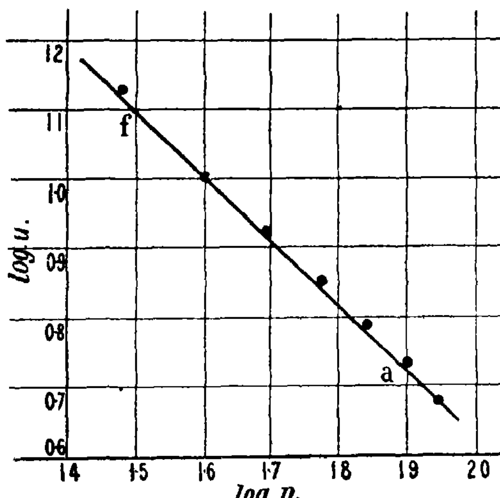

FIG 28

Find the best average values of the constants n and c for the range of values given

p	20	30	40	50	60	70	80	90
u	19·75	13·49	10·3	8·35	7·04	6·09	5·37	4·81
$\log p$	1·301	1·477	1·602	1·699	1·778	1·845	1·903	1·954
$\log u$	1·296	1·130	1·013	0·922	0·848	0·785	0·730	0·682

Plotting the values of $\log p$ and $\log u$ as in Fig. 28, a straight line is obtained; its equation may be written $\log p + n \log u = \log c$. To find the constants it is only necessary to substitute simultaneous values of $\log u$ and $\log p$ from the curve Thus at f, $\log p$ is 1 5, $\log u = 1{\cdot}1$, and at a, $\log p$ is 1·9, $\log u$ is 0 725. Substituting these values, we have

$$1\ 5 + n \times 1{\cdot}1 = \log c \qquad \text{(i)}$$

$$1\ 9 + n \times 0\ 725 = \log c \qquad \text{(ii)}$$

By subtraction, $0\ 4 = 0\ 375n$;

$$n = \frac{0\ 4}{0{\cdot}375} = 1\ 067.$$

Substituting this value for n in (i), we obtain the value of $\log c$;

$$1\ 5 + 1\ 067 \times 1{\cdot}1 = 2\ 6737;$$

$$c = 471\ 8$$

It will be noticed in the preceding example that the two varying quantities follow a somewhat complex law. In such

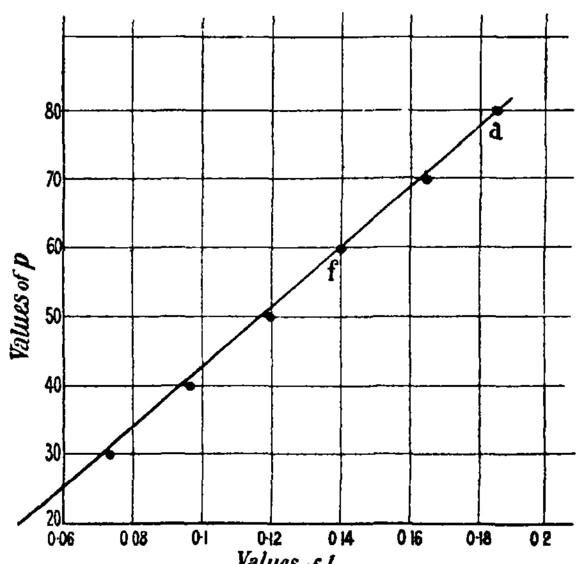

Fig 29

cases it is often possible to determine a simpler law, which between certain limits will give values closely approximating

to the correct ones And we may plot the reciprocals or squares, etc., of one instead of both the given quantities. Thus, in the preceding case, using the values of u, we may calculate values of the reciprocals $\frac{1}{u}$, and we obtain the following.

p	20	30	40	50	60	70	80	90
u	19 75	13 49	10 30	8 35	7 04	6 09	5·37	4·81
$\frac{1}{u}$	0·0506	0 0741	0 0971	0 120	0·142	0 164	0·186	0 208

Plotting p and $\frac{1}{u}$ as in Fig 29, a straight line may be drawn amongst the plotted points Its equation may be written

$$\frac{1}{u}=a+bp \quad \ldots\ldots\ldots \quad (\text{i})$$

At two points f and a in the line the values of p and $\frac{1}{u}$ are 60, 0·14, 80 and 0 185. Substituting these values in (i), we obtain

$$0{\cdot}185=a+80b$$
$$0\ 14\ =a+60b$$

Subtracting, $$0\ 045=\quad 20b\,;$$
$$b=0\ 00225$$

And, by substitution, a is found to be 0 005 Hence, the required relation is $\frac{1}{u}=0\ 005+0\ 00225\ p$ ....(i)

It will be seen that, for a given value of p, the value of $\frac{1}{u}$ or U can be obtained to a fair degree of accuracy Thus, (i) may be written

$$U=\frac{1}{0\ 005+0\ 00225p} \quad \ldots\ldots \quad (\text{ii})$$

The value of U when p is 80 is 5·37

From (ii) we obtain $U=\frac{1}{0\ 005+0\ 00225\times 80}=\frac{1}{0\ 185}$
$=5{\cdot}405.$

Hence, percentage error $=\frac{5\ 405-5\ 37}{5{\cdot}405}\times 100=0{\cdot}65.$

Ex. 10. Given that

$$y = 5\log_{10}x + 6\sin\frac{1}{10}x + 0{\cdot}084\ (x - 3{\cdot}5)^2 \ldots\ldots\ldots \quad (1)$$

Find a simpler function of x the values of which will have a small percentage error between $x=3$ and $x=6$.

Since the angle $\frac{1}{10}x$ is in radians, and as the values between 3 and 6 are required, we may use, for ease in calculation, numbers such as π, $\frac{3}{2}\pi$ and 2π for x

Thus, let $x=\pi$, then, by substituting in (1),

$$\begin{aligned} y &= 5\log\pi + 6\sin\frac{\pi}{10} + 0{\cdot}084\ (3{\cdot}1416 - 3{\cdot}5)^2 \\ &= 5\log\pi + 6\sin 18^\circ + 0{\cdot}084\ (-0{\cdot}3584)^2 \\ &= 5\times 0{\cdot}4972 + 6\times 0{\cdot}309 + 0{\cdot}0108 \\ &= 2{\cdot}486 + 1{\cdot}854 + 0{\cdot}0108 = 4{\cdot}351 \end{aligned}$$

In a similar manner, when x is $1{\cdot}5\pi$,

$$y = 5\ (\log 1{\cdot}5 + \log\pi) + 6\sin 27^\circ + 0{\cdot}084\ (1{\cdot}469);$$

$$y = 6{\cdot}214.$$

When $x=2\pi$,

$$y = 5\log 2\pi + 6\sin 36^\circ + 0{\cdot}084\ (7{\cdot}746) = 8{\cdot}168.$$

Plot these values and we find that a straight line can be drawn very evenly through the plotted points. Now, assume this simpler or linear function to replace the given one. Its equation may be written in the usual form,

$$y = ax + b \ldots$$

By substituting two pairs of simultaneous values, we can obtain the numerical values of the two constants a and b.

Thus,

$$\begin{aligned} 4{\cdot}15 &= 3a + b \\ 7{\cdot}5 &= 5{\cdot}75a + b \end{aligned}$$

Subtracting,

$$3{\cdot}35 = 2{\cdot}75a;$$

$$\therefore\ a = \frac{3{\cdot}35}{2{\cdot}75} = 1{\cdot}22.$$

Substituting this value,

$$4{\cdot}15 = 3\times 1{\cdot}22 + b;$$

$$\therefore\ b = 4{\cdot}15 - 3{\cdot}66 = 0{\cdot}49.$$

Hence, the simpler function required is $y = 1{\cdot}22x + 0{\cdot}49$.

It will be found on substitution that the values obtained from the simpler function are, for any value between the limits referred to, not more than 2 per cent. in error.

Ex. 11. At the following draughts in sea water, a particular vessel has the following displacements:

Draught h (feet)	15	12	9	6·3
Displacement T (tons)	2098	1512	1018	586
$\log h$	1·1761	1·0792	0·9542	0·7993
$\log T$	3·3218	3·1796	3·0076	2·7679

(i) Plot $\log T$ and $\log h$ on squared paper, and obtain a simple relation connecting T and h between the given limits.

(ii) If one ton of sea water measures 35 cubic feet, find the rule connecting V and h if V is the displacement in cubic feet.

Plotting $\log T$ and $\log h$, a straight line may be drawn lying evenly among the points.

The relation may be expressed by

$$ch = T^n, \qquad \text{(i)}$$

where c and n are constants.

To determine the numerical values of c and n, we may write the equation in the form

$$n \log T = \log c + \log h \qquad \text{(ii)}$$

From such a line we find that when $\log T$ is 3·0, $\log h$ is 0·95; and when $\log T$ is 3·3, $\log h$ is 1·153.

Substituting these values in (ii),

$$n \times 3{\cdot}3 = \log c + 1{\cdot}153 \qquad \text{(iii)}$$

$$n \times 3{\cdot}0 = \log c + 0{\cdot}95 \qquad \text{(iv)}$$

Subtracting, $\qquad 0{\cdot}3n = 0{\cdot}203;$

$$\therefore n = \frac{0{\cdot}203}{0{\cdot}3} = 0{\cdot}6767.$$

Substituting this value in (iv),

$$0{\cdot}6767 \times 3{\cdot}0 - 0{\cdot}95 = \log c = 1{\cdot}08 = \log 12;$$

$$\therefore c = 12.$$

Hence, (i) may be written in the form

$$T^{0{\cdot}6767} = 12h,$$

or

$$T = (12)^{\frac{1}{0{\cdot}6767}} \times h^{\frac{1}{0{\cdot}6767}}. \qquad \text{(v)}$$

This is not in a convenient form for calculation, hence we may write (v) in the form

$$T^2 = (12)^{\frac{2}{0 \cdot 6767}} \times h^{\frac{2}{0 \cdot 6767}},$$

and as $2 \div 0\ 6767 = 2{\cdot}955$ we may obtain a good approximation by using the nearest whole number 3 and adjusting the constant.

Thus, (v) may be written as

$$T^2 = b^3 h^3, \qquad \ldots \qquad \text{(vi)}$$

or

$$\frac{2}{3}\log T = \log b + \log h.$$

Hence, draw a line having a slope of $\frac{2}{3}$ and passing as evenly as possible through the points. To obtain the constant b, we have from (vi)

$$\log b = \frac{2}{3}\log T - \log h\ ;$$

at c, where $\log T$ is 3 4, $\log h$ is 1 227.

Substituting,

$$\log b = \frac{2}{3} \times 3\ 4 - 1{\cdot}227 = 1{\cdot}040\ ;$$

$$3 \log b = 3{\cdot}120 = \log 1318.$$

Hence, the relation is

$$T^2 = 1318 h^3.$$

Also,

$$V \div 35 = T\ ;$$

$$\left(\frac{V}{35}\right)^2 = 1318 h^3,$$

or

$$V^2 = 1615000 h^3.$$

Ex 12. In the following table some observed values of x and y are given:

x	0	1	2	3	4	5	6	7
y	0	0·7485	0·5988	0 5614	0 5444	0 5347	0 5284	0 5241

It will be noticed that as x increases, the corresponding values of y are decreasing. If the given points are plotted, a curve is obtained. To obtain the algebraic law connecting x and y—instead of $y = a + bx$, try

$$y = \frac{ax}{x - b}, \text{ or } x = a\frac{x}{y} + b. \qquad \ldots\ldots \qquad \text{(i)}$$

Values of x and calculated values of $\frac{x}{y}$ are as follows:

x	1	2	3	4	5	6	7
$\frac{x}{y}$	1·336	3·339	5·343	7·348	9·351	11·36	13·36

Values of x and $\frac{x}{y}$ are plotted in Fig. 30 and a line is drawn through the plotted points. To obtain the equation of the line, or to obtain the values of the constants, in (i), we may select two points f and a, and substitute in (i) the values of x and $\frac{x}{y}$. Thus,

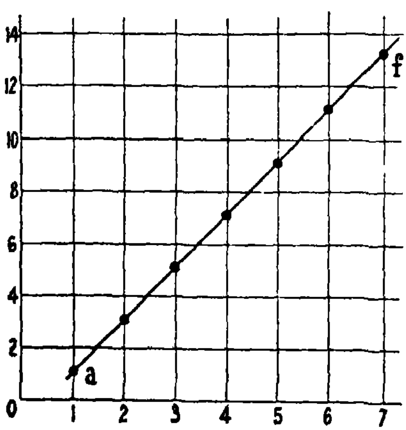

FIG. 30

$$7 = a \times 13·36 + b \qquad \text{(i)}$$
$$1 = a \times 1·336 + b \qquad \text{(ii)}$$
$$6 = 12·02a;$$
$$a = \frac{6}{12·02} = 0·5.$$

Substituting in (ii),

$$b = 0·33.$$

Hence, the relation connecting x and y is given by

$$x = 0·5\frac{x}{y} + 0·33,$$

or $\quad x = 0·5x + 0·33y$

Harmonic motion.—**A simple harmonic motion may be defined as the motion of the projection, upon a diameter, of a point moving uniformly in a circle.** Thus, let P be a point moving uniformly in a circle of radius a; the projections, M and N, of P on the axes move with simple harmonic motion.

Let ω denote the angular velocity of P (*i.e.* the angle in radians described by CP in one second). Then the angle ACP will be ωt; if x then denotes the displacement CM of the point M,

$$CM = x = a \cos \omega t. \quad \ldots\ldots\ldots \quad \text{(i)}$$

The **amplitude** is the greatest displacement on either side of the mean position C; hence, the amplitude is a, or it is the value of x when P is at A.

The **period** or **periodic time** is the interval of time taken by the point M to pass from A to A' and back again. It is usually denoted by letter T.

The **frequency**, f, is the reciprocal of the periodic time, or is $\frac{1}{T}$

It will be seen from Fig 31 that the motion of N is precisely similar to that of M, the difference being that when the displacement of M is a maximum that of N is a minimum, or zero The motion of points M or N is the simplest kind of vibrational motion, such as the up and down motion of a weight hanging from the end of a spiral spring, or the motion of the bob of a pendulum when the vibrations are small, or the motion of the prongs of a tuning fork, etc

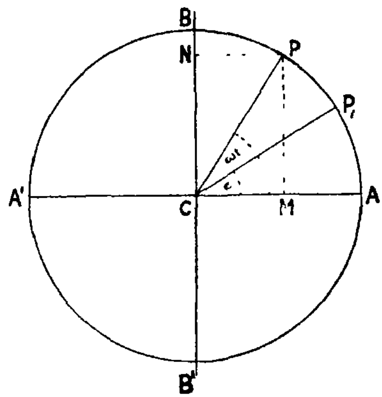

Fig 31

Similarly, the harmonic motion of point N may be expressed by

$$y = a \cos PCN = a \sin \omega t \quad \ldots \quad \ldots\ldots\ldots (ii)$$

It should be noticed that the coefficient a in (i) and (ii) gives in each case the amplitude, and ω the angular velocity of the corresponding circular motion

The period (T) is the time required for one revolution $= \frac{2\pi}{\omega}$, and the frequency f is the reciprocal of the periodic time, or $\frac{\omega}{2\pi}$;

$$\omega = 2\pi f$$

Ex. 1. $x = 7 \cos 3\omega t$.

This denotes a point moving with simple harmonic motion (usually denoted by the letters S.H M), the amplitude being 7, and angular velocity 3ω radians per second.

Ex. 2. Find the amplitude, angular velocity, period, and frequency of a point which has a simple harmonic motion given by the equation
$$x = 0.15 \cos 1.6t.$$

Comparing the terms in this with Eq. (i), the amplitude is found to be 0·15. The angular velocity is the coefficient of t, and is 1.6. The period is the time required for 1 revolution, and is therefore
$$\frac{2\pi}{1.6} = 3.927$$

The frequency is the reciprocal of the period,
$$f = \text{frequency} = \frac{1}{3.927} = 0.2546$$

If s denotes the distance of a moving point from its mid-position, at a time t, then if the relation between s and t is expressed by $s = a \sin qt$, or $s = a \sin 2\pi ft$, where f is the frequency, then the point is moving with S.H.M. of amplitude a.

The velocity (p. 337) $v = \dfrac{ds}{dt} = 2a\pi f \cos 2\pi ft$

The acceleration $\quad a = \dfrac{d^2s}{dt^2} = -4a\pi^2 f^2 \sin 2\pi ft\,;$

but $\quad s = a \sin 2\pi ft,$
$$a = \frac{d^2s}{dt^2} = -4\pi^2 f^2 s \quad \ldots\ldots \quad \text{(iii)}$$

Thus, the acceleration at any instant varies with and is directly proportional to the distance of the moving point from its mid-position, but in the opposite direction.

If M is the mass of a body $= W \div 32.2$, where W denotes the weight, then the force F acting towards the mid-position is given by
$$F = 4\pi^2 f^2 s \times M = \frac{4\pi^2 f^2 s \times W}{32.2} \quad \ldots\ldots \quad \text{(iv)}$$

Ex. 3. The relation between the distance s, from the middle point of the line of motion, being given by $s = A \sin(pt - e)$, where A, p and e are all constants. Find the velocity and acceleration at any instant.
$$v = \frac{ds}{dt} = Ap \cos(pt - e),$$
$$a = \frac{d^2s}{dt^2} = -Ap^2 \sin(pt - e).$$

Hence, as in the preceding case, the acceleration is equal to p^2 times the distance from a fixed point, this is the characteristic property of harmonic motion.

It should be carefully noticed from (iv) that f (the frequency) is squared Hence, when the frequency is doubled the force required is four times its former value, when the frequency is trebled the force is 9 times, etc

The sinuous curve corresponding to (i) could be set out on a horizontal base, equal distances denoting equal intervals of time and the various values of CM, or x, as ordinates

If the moving point starts at some point, say P_1 (Fig 32), and t_0 is the interval of time required from A to P_1, then the angle P_1CA may be written ωt_0 or e.

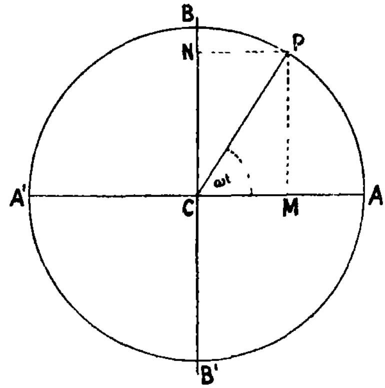

FIG. 32

Thus, when the point is at P the angle described is $\omega t+e$;

$$x=a\cos(\omega t+e) \quad \ldots . \quad \text{(v)}$$

The expression given by Eq (v) is found not only in engineering, but also in mathematical physics, more frequently than any other Every periodic function can be expressed in such terms, or a series of such terms. The most general form is expressed by Fourier's Theorem (p. 451).

It is more convenient in graphical work to project the various positions of point N (Fig 32).

For this purpose angles are more conveniently measured from the line BB', or 90° behind the initial line AA', and Equation (v), defining the successive positions of point N, becomes

$$y=a\sin(\omega t+e) \quad \ldots \ldots \quad \text{(vi)}$$

Other letters may be used instead of x, ω, etc Thus Eq. (vi) may be written

$$y=a\sin(bx+c). \quad \ldots \ldots \quad \text{(vii)}$$

Thus, when $x=0$, from Eq (vii)

$$y=a\sin c.$$

Or, when x is 0, the point P is ahead of the initial position B' by an angle $B'CP$ (Fig. 33).

The angle c when measured in a *positive* direction is usually called the angle of *lead* or *advance*; it is called the *lag* when measured in a *negative* direction.

It is of the utmost importance that the meanings attached to the constants a, b and c should be clearly made out.

Ex. 4. A point M has a simple harmonic motion in which the displacements x from the mid-position C is given in inches by

$$x = 2 \sin (1.5t + 0.4014) \qquad \text{(viii)}$$

Plot the curve and find the displacement of M when $t=0$ and also when t is 2 seconds.

From (viii), when $t=0$, $x = 2 \sin (0.4014)$

From Table VII. 0·4014 radians $= 23°$.

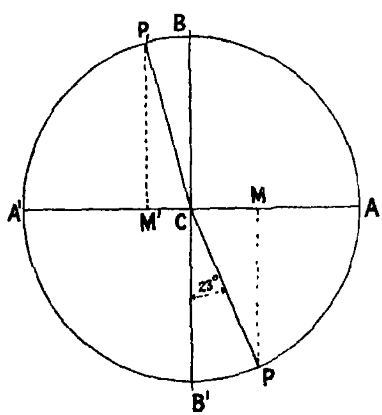

FIG. 33.

Hence, make the angle $B'CP = 23°$.

With C as centre describe a circle 2 inches radius, then P is the corresponding point in the auxiliary circle; and projecting on the diameter AA', the distance CM is the displacement required.

$\therefore CM = 2 \sin 23° = 2 \times 0.3907$;

$\therefore CM = x = 0{\cdot}7814$ inches

Similarly, when t is 2, we have, from (i),

$$x = 2 \sin (3 + 0.4014)$$
$$= 2 \sin (3{\cdot}4014)$$
$$= 2 \sin 194° \, 53';$$

$\therefore CM' = x = 0{\cdot}5$ inches

The following important theorem may be shown either graphically or analytically

A motion in a straight line which is compounded of two simple harmonic motions of equal periods and in the same straight line is itself a simple harmonic motion.

Thus, let two simple harmonic motions be expressed by the equations

$$a \sin (\omega t + e_1) \text{ and } b \sin (\omega t + e_2)$$

Add these, then we may write

$$A \sin(\omega t + e) = a \sin(\omega t + e_1) + b \sin(\omega t + e_2) \quad \ldots\ldots\ldots\ldots (\text{ix})$$

Expand the right-hand side of equation and rearrange,

$$A^2 = a^2 + b^2 + 2ab \cos(e_1 - e_2), \text{ also } \tan(e - e_1) = -\frac{b}{a};$$

$$\tan e = \frac{a \sin e_1 + b \sin e_2}{a \cos e_1 + b \cos e_2},$$

when $$e = 90^\circ, \tan e_1 = \frac{b}{a}.$$

Given the elements a, b, e_1 and e_2 of the component motion, the resultant motion can be obtained from (viii).

Ex 5. Given $a = 2$ inches, $b = 3$ inches, $e_1 = 0{\cdot}25$ radians, $e_2 = 1{\cdot}1$ radians. Determine graphically and measure the amplitude A and advance e of the resultant motion.

Also find and measure the displacement x when $t = 0$, and also when $t = 3$ seconds. The angular velocity ω is $\frac{1}{2}$ radian per second.

Substituting, the equation becomes

$$x = 2 \sin(\omega t + 0{\cdot}25) + 3 \sin(\omega t + 1{\cdot}1).$$

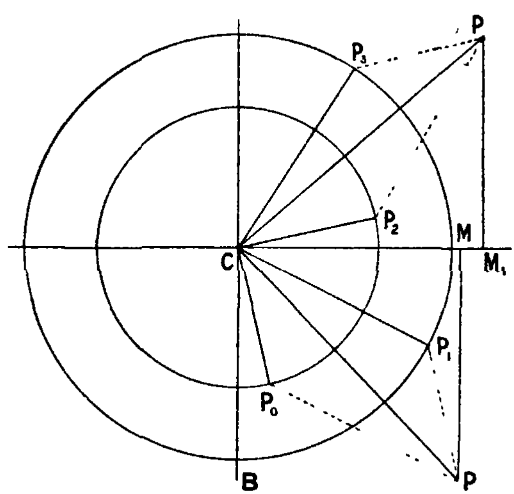

FIG. 34

With centre C (Fig. 34) draw two circles of radii 2 and 3 inches respectively. When $t = 0$, the first component motion gives angular advance $= 0{\cdot}25$ radian $= 14^\circ\ 19'$. The second component gives angular

advance $=1\cdot1$ radian $=63^\circ\ 1'$. Hence, make the angle $BCP_0=14^\circ\ 19'$, and the angle $BCP_1=63^\circ\ 1'$, giving two points, P_0 on the smaller and P_1 on the larger circle respectively.

Complete the parallelogram of which P_0C and P_1C are adjacent sides; then CP, equal to 4 55 inches, is the amplitude and $B'CP$ is the angle of advance equal to $43^\circ\cdot5$. Projecting P on to CA the displacement CM is found to be 3 15 in

Again, when t is 3 seconds

Substituting the given angular velocity, the equation becomes

$$x=2\sin(0\cdot5t+0\ 25)+3\sin(0\cdot5t+1\ 1),$$

and this, when $t=3$, gives for the first component an angle of advance of

$$0\cdot5\times3+0\cdot25=1\ 75 \text{ radians}=100^\circ\ 16' \text{ very nearly,}$$

and for the second component

$$3\times0\ 5+1\ 1=2\ 6 \text{ radians}=148^\circ\cdot98$$

Set off the angle BCP_2 equal to $100^\circ\ 16'$, and the angle BCP_3 equal to $148^\circ\ 58'$, giving as before the two sides of a parallelogram. Completing the parallelogram, CP is the resultant amplitude and BCP the angle of advance, giving 4·55 inches for the former and 131° for the latter. The displacement CM' obtained by projection is 3·45 in It will be noticed that the parallelogram CP_2PP_3 is merely the parallelogram CP_0PP_1 in another position Or, in other words, the resultant motion will be as though the parallelogram CP_0PP_1 were to rotate as a rigid framework attached to C, and made to move about C as a centre All positions of P will therefore lie on a circle centre C and radius CP.

When the numerical values of the constants a, b, and c are known, the curve, or graph, corresponding to $y=a\sin(bx+c)$ may be set out. Then, for assumed values of x corresponding values of y may be calculated, and the curve passing through the plotted points obtained

As $bx+c$ denotes the angle in radians it will simplify the arithmetical work if b be taken to be a multiple or sub-multiple of π. Hence, let $b=\dfrac{10}{57\cdot3}$, and let c be $\dfrac{\pi}{6}=30^\circ$.

If the amplitude a be 2·5, then we have the necessary data, as in the following example ·

Ex. 6. Plot the curve $y=2\cdot5\sin\left(\dfrac{10x}{57\cdot3}+\dfrac{\pi}{6}\right)$

Values of y corresponding to various values of x may be found. Thus, when $x=4$, $\sin(40^\circ+30^\circ)=\sin 70^\circ$;

$$\therefore\ y = 2{\cdot}5 \sin 70^\circ = 2{\cdot}5 \times 0\ 9397$$
$$= 2\ 349.$$

Other values of y may be tabulated as follows ·

t	0	1	2	3	4	5	6	7	8	9
y	1 25	1·607	1·915	2·165	2 349	2 462	2 5	2·462	2·349	2 165

From these values the curve may be plotted The sinuous line is much more easily obtained by graphical construction, as on p. 145.

The graph of $y = Ae^{kt}$, or $y = Ae^{kx}$—when the constants A and k are known and e is the base of Naperian logarithms $= 2{\cdot}718$—can be obtained by assuming various values for x or t and calculating corresponding values of y.

Ex. 7. Plot the curve

$$y = Ae^{kx},$$

when $A = 1$, $k = 0{\cdot}3$.

Substituting the given values, the equation becomes

$$y = e^{0\ 3x}.$$

Assuming values 0, 1, for x, values of y can be calculated.

Thus, let $x = 4$, then

$$y = 2{\cdot}718^{1\cdot 2},$$

or $\log y = 1{\cdot}2 \log 2\ 718$
$= 1\ 2 \times 0\ 4343$
$= 0{\cdot}52116$;

$\therefore\ y = 3{\cdot}321$

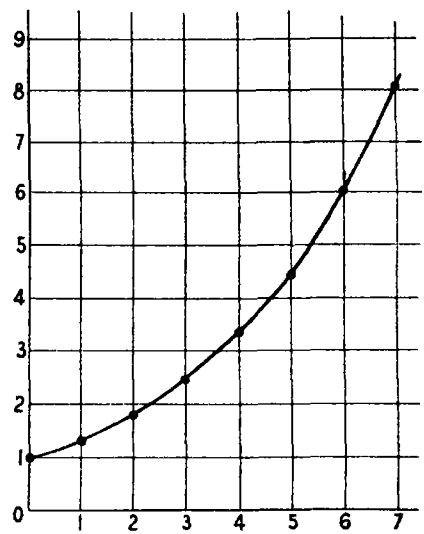

FIG. 35 —Graph of $y = e^{0\ 3x}$

In a similar manner other values of y can be ascertained as follows:

x	0	1	2	3	4	5	6	7	8
y	1	1 35	1 822	2·460	3·321	4·481	6·049	8·166	11·03

A portion of the curve is drawn in Fig. 35

Damped oscillations.—A simple experimental apparatus illustrating what is meant by damped oscillations may consist of a comparatively heavy cylindrical disc suspended at one end of a wire. The other end of the wire is fixed to a suitable support, and the disc may be made to oscillate in a liquid such as water, oil, glycerine, etc.

When displaced from its position of rest and allowed to oscillate freely the amplitude of the oscillation diminishes more or less rapidly, due to the viscosity of the liquid.

If on a base denoting intervals of time, ordinates of the curve denote amplitudes, then the maximum amplitude is obviously at a time $t=0$, and therefore equal to A, and the amplitudes in successive swings diminish according to the logarithmic law $s=Ae^{-kt}$

Thus, a steel wire may be fastened at one end to a fixed support, and at the other to a comparatively heavy disc of metal, a pointer fixed to the wire can be displaced through any convenient angle as indicated on a graduated disc Then when released, the pointer will oscillate backwards and forwards, through its position of equilibrium, with logarithmic decaying amplitude.

If s is the displacement, or amplitude, of point p at the time t, then the law connecting displacements separated by equal times of one period is given by $s=Ae^{-kt}$. .... (i)

The numerical values of the two constants A and k are readily obtained. Thus, let the pointer p be displaced through (say) an angle of 180° ; if this denotes the time $t=0$, then from (i), when $t=0$, we have

$$180^\circ = Ae^0,$$

or

$$A = 180^\circ.$$

At the instant the pointer is released, let a stop-watch be started. Then the time of successive oscillations and the amplitudes can be read off; these may be tabulated. Similar observations should be made when different fluids, water, oil, glycerine, etc, are used

Eq. (i) can be written $\log s = \log A - kt \log e$.

Plotting t and $\log s$ as co-ordinates of points, the points will be found to lie on a straight line, and the values of k, which

will express the *relative viscosities* of the liquids, can be obtained.

The relation between s and t is given by the differential equation (see p. 480)

$$\frac{d^2s}{dt^2}+2k\frac{ds}{dt}+s=0,$$

The solution is $s=ae^{-kt}\sin\{(\sqrt{1-k^2})t+b\}$,

where a and b are constants to be determined (p. 480).

Values obtained from an experiment are given

Water			Oil.			Glycerine		
s	$\log s$	t	s	$\log s$	t	s	$\log s$	t
180°	2 255	0	180°	2·255	0	180°	2 255	0
164°	2 215	39	92°	1·964	41·4	24°	1 380	14·4
149°	2 173	79	46°	1 663	83 6	3°	0·477	28·4
137°	2 137	119	22°	1 342	126·2	0 7°	0 155	42 2
125°	2 097	159	10°	1·0	169·0			
115°	2 061	198	5°	0 699	210·6			

The values of the constants may be obtained by plotting, and the relations become·

For water, $s=180e^{-0\ 0023t}$; for oil, $s=180^{-0\ 0162t}$; for glycerine, $s=180e^{-0\ 14t}$

These values should be verified Thus, in the case of glycerine, let $t=14{\cdot}4$, and proceed to find the value of s.

$$\log_e s=\log A-k\times t\log_e e,\text{ or to base 10,}$$

$$\log_{10}s=\log_{10}180-0{\cdot}14\times14{\cdot}4\times0{\cdot}4343$$

$$=2{\cdot}2553-0{\cdot}14\times14{\cdot}4\times0{\cdot}4343.$$

The product can be obtained either by a slide rule or logarithms. Thus, if x denote the product,

$$\log x=\log 0{\cdot}14+\log 14{\cdot}4+\log 0{\cdot}4343=\bar{1}{\cdot}9413;$$

$$\therefore\ x=0{\cdot}8737,$$

$$\therefore\ \log s=2{\cdot}2553-0\ 8737=1{\cdot}3816.$$

In a similar manner the remaining two values may be verified.

Graph of $y=Ae^{kx}\sin(bx+c)$.—One of the most important curves in engineering is given by the equation

$$y=Ae^{kx}\sin(bx+c)$$

When k is negative this equation indicates a damped vibration.

It will be noticed that this curve is a combination of the two preceding curves, *i.e.* $y=e^{kx}$ and $y=A\sin(bx+c)$, and, if plotted on the same sheet, it is only necessary to multiply together the ordinates, for the same value of x, of the two curves to obtain the ordinates of the new curve.

Ex. 8 In the equation $y=Ae^{kx}\sin(bx+c)$.

Let $A=2\ 5$, $k=0\ 3$, $b=\dfrac{10}{57\ 3}$, $c=\dfrac{\pi}{6}$

Calculate values of y for values 0, 2, 16 for x, and plot the curve Find the slope of the curve at the point $x=4$

Substituting the given values, the equation becomes

$$y=2\ 5e^{0\ 3x}\sin\left(\frac{10x}{57\ 3}+\frac{\pi}{6}\right) \qquad \ldots (1)$$

Substituting values 0, 2, 4 for x, then, from (1), corresponding values of y can be obtained, or values may be obtained from Exs. 5 and 6. Thus, when $x=4$ the ordinate of the curve $y=e^{0\ 3x}$ is 3 321

For the same value of x the ordinate of the curve

$$y=2\ 5\sin\left(\frac{10}{57\cdot3}x+\frac{\pi}{6}\right) \text{ is } 2\cdot349.$$

The product will give the ordinate of the new curve ;

$$i\ e\quad 3\cdot321\times2\cdot349=7\cdot8$$

By substituting values of x, corresponding values of y can be calculated Thus, let $x=4$, then, substituting in (1),

$$y=2\cdot5e^{1\cdot2}\sin(70^\circ)$$
$$=2\ 5e^{1\cdot2}\times0\ 9397.$$
$$\log y=\log2\cdot5+1\cdot2\log e+\log 0\ 9397$$
$$=0\cdot3979+0\cdot5212+\bar{1}\ 9730=0\cdot8921\ ;$$
$$\therefore\ y=7\cdot8.$$

Other values of x can be assumed and values of y calculated as follows ·

x	0	2	4	6	8	10	12	14	16
y	1·25	3·49	7·8	15·12	25·89	38·46	45·75	42·24	- 52·72

The curve passing through the plotted points may be obtained as in Fig. 36.

To find the slope at $x=4$, we must find $\frac{dy}{dx}$ (p 303).

When $y=Ae^{kx}\sin(bx+c)$,

$$\frac{dy}{dx}=Aae^{kx}\sin(bx+c)+Abe^{kx}\cos(bx+c)$$
$$=2\cdot5\times0\cdot3e^{1\cdot2}\sin70^\circ+2\cdot5\times\frac{10}{57\cdot3}e^{1\cdot2}\cos70^\circ$$
$$=2\cdot8453.$$

Ex. 9 Obtain the graphs of the following:

$$\text{(i)}\ y_1=2\cdot5\sin\left(\frac{10x}{57\cdot3}+\frac{\pi}{6}\right), \qquad \text{(i)}$$

$$\text{(ii)}\ y=2\cdot5e^{-0\cdot02x}\sin\left(\frac{10x}{57\cdot3}+\frac{\pi}{6}\right)$$

(i) This graph may be obtained, as in preceding cases, by calculation; but, more easily, by graphical construction, as follows

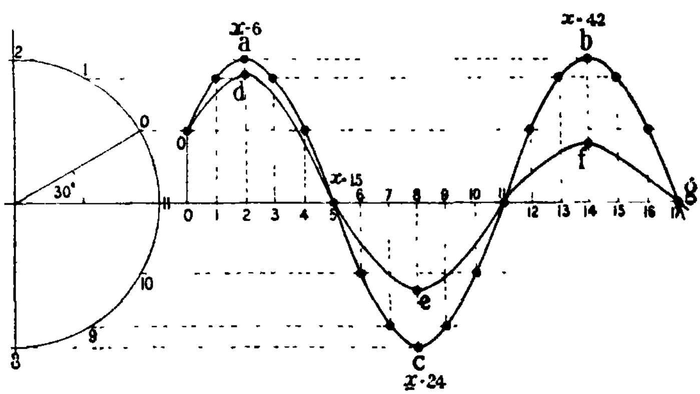

FIG 36 —Graph of $y=2\cdot5e^{-0\cdot02x}\sin\left(\frac{10x}{57\cdot3}+\frac{\pi}{6}\right)$

Draw a circle 2·5″ radius and divide its circumference into 12 equal parts. Now draw a straight line (Fig. 36) to denote the periodic time, or time taken by the point to move once round the circle. Divide the line into the same number of parts as the circle, *i.e.* into 12 equal parts, and at each point set up ordinates. Then points in the curve may be obtained by projection. In this

manner, by drawing a curve through the points, a curve such as *oacbg* (Fig 36) is obtained

The sine of an angle has its maximum positive values when the angle is 90°, or of the form $(4n+1)90°$ Similarly, the maximum negative values occur when the angle is 270°, *i e* of the form $(4n+3)90°$, the maximum positive values occurring at a and b, where $x=6$ and $x=42$, and maximum negative value at c, where $x=24$

(ii) For various values of x, values of $y_2=e^{-0\ 02x}$ may be calculated and tabulated. Thus, when $x=6$, $y_2=e^{-0\ 12}=0\ 8869$ Multiplying these by the corresponding values of y_1, we obtain the ordinate required as in the following table.

x	6	15	24	33	42
y_1	2 5	0	−2 5	0	2 5
y_2	0 8869	0 7408	0·6187	0 5169	0 4317
$y_1\times y_2=y$	2 2172	0	−1 5467	0	1 0792

As (ii) may be written in the form

$$y=2\ 5\sin\left(\frac{10x}{57\ 3}+\frac{\pi}{6}\right)\times e^{-0\ 02x},$$

it follows that numerical values of y, when various values are assumed for x, can be obtained from the product of corresponding terms y_1 and y_2; these are given in the last column of the above table

Plotting values of x and y, the curve *odfg*, passing through the plotted points, and known as a *damping curve*, is obtained

Equations of the form $\mathbf{T=a+bv^n}$.—When the relation between the variables T and v involves three constants some transformation, as in the following example, is necessary before the constants can be determined

Ex 10. Two variable quantities v and T are supposed to be connected by a relation of the form

$$T=a+bv^n \qquad \text{(i)}$$

When v is 3, T is 48 97; when v is 4, $T=41·49$; and when v is 6, T is 34·74.

Determine the numerical values of the three constants a, b, and n Also find T when v is 5.

We may denote the three given values of v by v_1, v_2, and v_3; and the corresponding values of T by T_1, T_2, and T_3, respectively.

As Eq. (i) is not adapted to logarithms, it may be written

$$T - a = bv^n.$$

Writing T_1 for T, and v_1 for v, and take logarithms of both sides,

$$\log(T_1 - a) = \log b + n \log v_1. \quad \ldots \quad \text{(ii)}$$

Similarly $$\log(T_2 - a) = \log b + n \log v_2. \quad \ldots \quad \text{(iii)}$$

Subtracting (iii) from (ii),

$$\log(T_1 - a) - \log(T_2 - a) = n(\log v_1 - \log v_2). \quad \ldots \quad \text{(iv)}$$

In a similar manner we obtain

$$\log(T_1 - a) - \log(T_3 - a) = n(\log v_1 - \log v_3). \quad \ldots \quad \text{(v)}$$

Dividing (iv) by (v),

$$\frac{\log(T_1 - a) - \log(T_2 - a)}{\log(T_1 - a) - \log(T_3 - a)} = \frac{\log v_1 - \log v_2}{\log v_1 - \log v_3}. \quad \ldots \quad \text{(vi)}$$

Thus, we obtain an equation involving only the constant a, which has to be determined

Eq (vi) may be written

$$f(a) = \frac{\log(T_1 - a) - \log(T_2 - a)}{\log(T_1 - a) - \log(T_3 - a)} - \frac{\log v_1 - \log v_2}{\log v_1 - \log v_3}. \quad \ldots \quad \text{(vii)}$$

The solution required being that for which $f(a) = 0$.

Substituting various values for a in (vii), the value of a, which makes the expression zero, and therefore satisfies the given equation, can be obtained. Thus, let $a = 20$.

$$f(a) = \frac{\log(48 \cdot 97 - 20) - \log(41 \cdot 49 - 20)}{\log(48 \cdot 97 - 20) - \log(34 \cdot 74 - 20)} - \frac{\log 3 - \log 4}{\log 3 - \log 6}$$

$$= \frac{\log 28 \cdot 97 - \log 21 \cdot 49}{\log 28 \cdot 97 - \log 14 \cdot 74} - \frac{\log 3 - \log 4}{\log 3 - \log 6}$$

$$= \frac{1 \cdot 4620 - 1 \cdot 3322}{1 \cdot 4620 - 1 \cdot 1685} - \frac{0 \cdot 4771 - 0 \cdot 6021}{0 \cdot 4771 - 0 \cdot 7782}$$

$$= \frac{0 \cdot 1298}{0 \cdot 2935} - \frac{0 \cdot 1250}{0 \cdot 3011};$$

$$\therefore f(a) = 0 \cdot 4424 - 0 \cdot 4152 = 0 \cdot 0273$$

Substitute values 10, and 30, for a, then corresponding values of $f(a)$ may be obtained and tabulated as follows:

a	10	20	30
$f(a)$	0·0540	0·0273	−0·0536

As the value of $f(a)$ changes sign in passing from $a=20$ to $a=30$, it follows that the value of a which will make $f(a)$ equal to zero lies between the two values. By plotting, or by trial, a is found to be 25.

Hence, the given equation may be written

$$T=25+bv^n.$$

and substituting in Eq. (iv), we find

$$n=-1{\cdot}3,$$

again, from Eq. (ii) or Eq. (iii)

$$b=+100.$$

Hence, the required relation is

$$T=25+100v^{-1{\cdot}3}.$$

Substituting for v, the given value

$$T=25+100\times 5^{-1{\cdot}3}=37{\cdot}34.$$

In some cases, by assuming a value for $n=2$, 3, etc., it is possible to obtain the law, as in the following example.

Ex. 11. A series of values of two variables (which may be denoted by x and y) are given in the following table. Find the relation between x and y.

x	0	1	2	3	4	5	6	7
y	2·35	2·77	4·03	6·13	9·1	12·85	17·5	22·9

When these values are plotted a curve passes through the plotted points; but, by plotting values of y and x^2, a straight line is obtained. Its equation may be written $y=a+bx^2$, and by substitution the constants a and b are found to be 2·35 and 0·42. Hence, the required relation is

$$y=2{\cdot}35+0{\cdot}42x^2$$

It will be obvious that the same result would have been obtained by using the general formula

$$y=a+bx^n.$$

Instead of the preceding, a still more general formula may be used, viz.:

$$y=a+b\ 10^{cx}, \qquad \text{(i)}$$

and from three given values of x and y the values of the three constants a, b, and c, may be found.

Thus, if three given values of x and y are denoted by x_1, x_2, and x_3, and the corresponding values of y by y_1, y_2, and y_3, respectively, then substituting in (i)

$$\log(y_1-a)=\log b+cx_1\log 10;$$

but as $\log_{10} 10 = 1$, we obtain using common logarithms

$$\log(y_1 - a) = \log b + cx. \quad \ldots\ldots \quad \ldots \text{(ii)}$$

Hence, as in the preceding cases,

$$\log(y_1 - a) - \log(y_2 - a) = c(x_1 - x_2) \quad \ldots\ldots\ldots\ldots\text{(iii)}$$

and $$\log(y_1 - a) - \log(y_3 - a) = c(x_1 - x_3),$$

and by division

$$\frac{\log(y_1 - a) - \log(y_2 - a)}{\log(y_1 - a) - \log(y_3 - a)} = \frac{x_1 - x_2}{x_1 - x_3}.$$

From this result the value of the constant a can be obtained, and by substitution in (ii) and (iii) the values of the remaining two constants b and c are found

Ex. 12. In how many years will a sum of money double itself at r per cent per annum?

Let A denote the amount, P the sum of money, and n the number of years.

Then $$A = P\left(1 + \frac{r}{100}\right)^n. \quad \ldots\ldots\ldots\ldots \text{(i)}$$

When $A = 2P$, then from (i)

$$2P = P\left(1 + \frac{r}{100}\right)^n,$$

or $$n = \log 2 \div \log\left(1 + \frac{r}{100}\right). \quad \ldots\ldots \text{(ii)}$$

Taking various values 2, $2\frac{1}{2}$, 3, , etc., for r we may calculate n in each case from (ii)

If the values of r and n are plotted a curve can be drawn through the points; plotting n and $\frac{1}{r}$ the points are found to lie nearly on a straight line; and when r does not exceed 5, it will be found that the approximate relation

$$n = 70 \div r \quad \ldots \quad \text{(iii)}$$

may be used

Thus, if $r = 5$, then from (ii)

$$n = \log 2 \div \log\left(1 + \frac{5}{100}\right)$$

$$= \frac{\log 2}{\log 1{\cdot}05} = 14{\cdot}2$$

Hence, a sum of money at 5 per cent. per annum will double itself in 14·2 years.

Using the approximate relation given by (iii), we have

$$n = \frac{70}{5} = 14 \text{ years.}$$

EXERCISES. XV.

1. The following observed values of E and R are supposed to be related by a linear law $R=a+bE$, but there are errors of observation. Find by plotting the values of R and E the most probable values of a and b.

(i)

E	2·5	3·5	4·4	5·8	7·5	9·6	12·0	15·1	18·3
R	13·6	17·6	22·2	28·0	35·5	47·4	56·1	74·6	84·9

(ii)

E	5	9·44	13·37	15·56	21·94	26·12	30·25
R	14	28	42	56	70	84	98

(iii)

E	1	1·84	2·75	3·62	4·56	5·4	6·18
R	14	28	42	56	70	84	98

(iv)

E	7	8·5	10	11·5	13·25	14·75	16·25	17·75
R	27·9	41·9	55·9	69·9	83·9	97·9	111·9	125·9

2. The relation between two variable quantities F and R is given by $F=cR+d$.

If when R is 20, F is 140, and when R is 50, F is 395. Find the numerical values of c and d.

3. The expression $ax^2+bx-30$ is equal to 240 when $x=5$, and equals 100 when $x=-2$. Find its value when $x=11$.

4. If $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$, and $a=5$; then taking various values of ϕ between 0 and, say 1·5, calculate x and y, and plot this part of the curve.

5. Plot the curve $y=\sin x$.

Give x values which are multiples and sub-multiples of $\frac{\pi}{2}$.

Notice that y is a maximum when $x=\frac{1}{2}\pi$, $\frac{5}{2}\pi$, etc., and y is a minimum when $x=-\frac{1}{2}\pi$, $\frac{3}{2}\pi$, etc.

6 The following values of p and u, the pressure of specific volume of steam, are taken from tables.

p	15	20	30	40	50	65	80	100
u	25·87	19·72	13·48	10·29	8·34	6·52	5·37	4·36

Find whether an equation of the form $pu^n = \text{const.}$ represents the law connecting p and u, and if so, find the best average value of the index n for the range of values given.

7 Values of p and u are given in the following table. Find the best average values of n and c in the equation $pu^n = c$ for the range of values given.

p	100	91·3	84·5	78·23	68·71	67·85	63·54	56·19
u	3·0	3·2	3·4	3·6	3·8	4·0	4·2	4·6

Also find the value of p when $u = 4·4$.

8. A series of values of x and y are given in the following table; assuming that the relation between x and y can be expressed by $y = a + bx^2$. Find the numerical values of the constants a and b

x	0	1	2	3	4	5	6	7	8
y	3·25	3·45	3·65	5·0	6·45	8·25	10·45	12·0	16·0

9. A series of values of H and Q are given in the following table. Try if the relation between H and Q can be expressed in the form $Q = cH^n$. If so, obtain the best average values of the constants c and n

H	1·0	1·5	2·0	2·5
Q	2·6	7·3	14·9	26·0

10 The following table gives the ordinates of a curve at various distances (x) measured from one end of the axis. Find the mean ordinate and area of the curve.

Ordinates	53	75	84	94·5	123	139	134	106	76	45
x inches	0	9	22	41	62	78	97	114	128	144

11. The following values of x and y are supposed to be related by an equation of the form $y=a+bx^2$. Plot on squared paper, and find numerical values of a and b

x	0	1	2	3	4	5	6	7	8
y	2	2·05	2·2	2·45	2·8	3·25	3·8	4·45	5·2

12. Two variables x and y are connected by the relation $y=a+bx+cx^2$; the following simultaneous values of x and y are given. Find the numerical values of a, b and c.

x	0	1	2	3	4	5	6	7	8
y	2	1·85	1·8	1·85	2·0	2·25	2·6	3·04	3·6

13. Two variable quantities x and y are found to be related to one another for certain values of x as shown in the following table:

x	2	3	4	5	6
y	6·9	11·2	15·7	20·7	25·8

Try if the quantities are connected by a law of the form $y=ax^n$; and if so, find approximately the values of n and a

14 The following quantities are thought to follow a law like $pv^n=$ constant. Try if they do so, find the most probable value of n.

p	1	2	3	4	5
v	205	114	80	63	52

15. Taking $x=0, 1, \ldots 5$, find values of y if

$$y=\frac{2\cdot5x}{3+0\cdot5x},$$

and draw the curve.

16. Plot the following observed values of A and B on squared paper, and determine the most probable law connecting A and B. Then assume this law to be correct and find the percentage error in the observed value of B when A is 150

A	0	50	100	150	200	250	300	350	400
B	6·2	7·4	8·3	9·5	10·3	11·6	12·4	13·6	14·5

17. The following values of x and y are connected by a relation of the form $y = ax^2 + b$. Find the numerical values of the constants a and b and the area of the curve from $x=0$ to $x=8$.

x	0	1	2	3	4	5	6	7	8
y	2.5	2.8	3.7	5.2	7.3	10	13.3	17.2	21.7

18. The following values of x and y are connected by a relation of the form $y = Ae^{bx}$. Find the numerical values of the constants A and b.

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y	0.4524	0.4093	0.3704	0.3352	0.3032	0.2744	0.2483	0.2246	0.2033	0.1839

19. Work the following three exercises as if in each case one were alone given, taking in each case the simplest supposition which your information permits.

(*a*) The total yearly expense in keeping a school of 100 boys is £2100; what is the expense when the number of boys is 175?

(*b*) The expense is £2100 for 100 boys, £3050 for 200 boys; what is it for 175 boys?

(*c*) The expenses for three cases are known as follows:

£2100 for 100 boys,
£2650 for 150 boys,
£3050 for 200 boys.

What is the probable expense for 175 boys?

If you use a squared paper method, show all three solutions together.

20. A steam electric generator is found to use the following amounts of steam per hour for the following amounts of power:

Lbs. of steam per hour	4020	6650	10800
Indicated horse-power	210	480	706
Kilowatts produced	114	290	435

Find the indicated horse-power and the weight of steam used per hour when 330 kilowatts are being produced.

21. (i) Given $T_1=28{\cdot}689$, $T_2=28{\cdot}249$, $T_3=27{\cdot}546$, $v_1=2150$, $v_2=1900$, $v_3=1600$ The relation between T and v may be expressed by $T=a+bv^n$. Calculate the numerical values of a, b and n

(ii) If the relation is $T=a+bv^{-1}$, find the values of a and b which will make the formula best represent the observations

22. Experiments on the loss of head in a lead pipe of 0·4 inches diameter give, for a length of $3\frac{1}{2}$ feet, the following results ·

Velocity of flow in feet per second $=v$	8·04	11·67	14 43	17 41	19·9
Observed difference of head in feet of water $=h$	3 03	6 11	9 07	12 21	15 62

Test whether the results can be expressed by a formula of the type $h \propto v^n$, and if so, obtain the value of n If we assume that

$$h=f\frac{4l}{d}\frac{v^2}{2g},$$

in which the length l and diameter d of the pipe are in feet, what is the best value of the coefficient f? Take $g=32{\cdot}2$

23. A is the horizontal sectional area of a vessel in square feet at the water level, h being the vertical draught in feet

A	14850	14400	13780	13150
h	23·6	20 35	17 1	14 6

Plot; and read off values of A for values of $h=23$, 20, 16

If the vessel changes in draught from 20 5 to 19 5, what is the diminution of its displacement in cubic feet?

24. A series of values of v and T are given in the following table Assuming the relation between T and v to be given by $T=a+bv^n$, find the numerical values of the constants a, b and n

T	2 867	2·876	2·884	2 891	2 899	2 906
v	3 0	3·2	3·4	3·6	3·8	4·0

25. Two variables S and v are assumed to be connected by a relation of the form $S=c+av^n$. Three values of v are 2 8, 3 4 and 4·0, and the corresponding values of S are 7 858, 7·88 and 7 9 respectively; find the numerical values of the constants c, a and n.

26. The slide value of a horizontal steam-engine derives its motion from a point P in a link A_1A_2, where $A_1P=\frac{1}{3}A_1A_2$.

The horizontal displacements of A_1 and A_2 for any crank position θ are given by the equations

$$x_1=2\cdot5''\sin(\theta+27^\circ),\quad x_2=2\cdot6''\sin(\theta+150^\circ)$$

The resulting motion of the value being defined by the equation

$$x=a''\sin(\theta+a),$$

find the half travel a'' and the advance a

27 A series of soundings taken across a river channel is given by the following table, x being the distance in feet from one shore and y the corresponding depth in feet:

x	0	10	16	23	30	38	43	50	55	60	70	75	80
y	10	20	26	28	30	31	28	23	15	12	8	6	0

Find the area of the cross-section.

28. If $x=a\sin pt+b\cos pt$ where a, b and p are constants Show that this is the same as $x=A\sin(pt+e)$ if A and e are properly evaluated, and find the values of A and e

29. The relation between s, the space described by a moving body, and t, the time in seconds, is given by

$$s=Ae^{-kt}\cos 2\pi\left(\frac{t}{t_1}+e\right).$$

Show that its velocity at time t is (p. 337).

$$\frac{ds}{dt}=-Ae^{-kt}\left\{\frac{2\pi}{t_1}\sin 2\pi\left(\frac{t}{t_1}+e\right)+k\cos 2\pi\left(\frac{t}{t_1}+e\right)\right\}$$

30 Given $y=2\cdot45e^{0\cdot4x}$ calculate y for each of the following values of x, and plot the curve.

x	0	1	2	3	4	5	6	7	8
y									

Find the slope of the curve at the point $x=4$, also the average value of y from $x=0$ to $x=8$

31. Plot the curve $y=3\sin x+4\cos x$, and from your curve see that the figure obtained is really a sine curve with different constants.

32. Plot the curve $y=25e^{-0\cdot1x}\sin(bx+c)$, where $b=\frac{10}{57\cdot3}$, $c=\frac{\pi}{6}$.

33. A body weighs 1610 lbs., the force F in lbs. necessary to raise it a distance x feet is automatically recorded, and is as follows:

x	0	11	20	34	45	55	66	76
F	4010	3915	3763	3532	3366	3208	3100	3007

Find the work done on the body when it has risen 70 feet? What is then the velocity of the body?

34 A car weighs 10 tons; what is its mass in engineers' units? It is drawn by the pull P lbs, varying in the following way, t being seconds from the time of starting

P	1020	980	882	720	702	650	713	722	805
t	0	2	5	8	10	13	16	19	22

The retarding force of friction is constant and equal to 410 lbs *Plot $P-410$ and the time t, and find the time average of this* excess force What does this represent when it is multiplied by 22 seconds? What is the speed of the car at the time 22 seconds from rest?

CHAPTER VIII.

SOLUTION OF TRIANGLES.

Solution of triangles.—In every triangle there are **six elements**, viz the **three angles** and the **three sides**. To solve a triangle, three of these elements must be known—one at least of these being a side. The angles are denoted by the letters A, B, C, (Fig. 37), at each angular point. The angle ACB, for example, is simply referred to as the angle C. The side AB opposite the angle C is denoted by the letter c, and similarly the other two sides of the triangle by a and b.

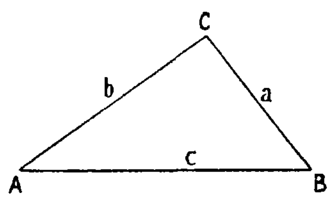

FIG 37

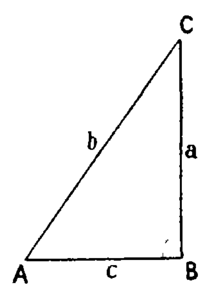

FIG. 38

When the angle B (Fig 38) is a right angle, the three sides are connected by the relation $b^2=a^2+c^2$

It is advisable in the solution of triangles to have some convenient method of checking the results obtained This check is furnished by drawing the triangle on *squared paper*, using the sides of the squares as suitable units of length, and setting out angles by means of (a) chords of angles (Table VIII.); (b) a table of tangents (Table VI.); or (c) a protractor

It may be difficult to measure with sufficient accuracy by graphical methods, hence, the magnitudes of lines, or angles, are most conveniently obtained by calculation. Various formulae adapted to logarithmic computation, together with the tables of ratios of angles (IV., V and VI), are used for the purpose.

The remaining elements of a triangle may be obtained either by construction or by calculation when the data include :—

(a) **Two sides and an angle.**
(b) **The three sides.**
(c) **Two angles and one side.**

The following formulae may be used ·

$$\text{(i)} \quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

or

$$\frac{a}{b}=\frac{\sin A}{\sin B}.$$

The sum of the three angles of a triangle are equal to 180°, so that when A and B are known, C is also known

$$\text{(ii)} \quad a^2=b^2+c^2-2bc\cos A,$$

or,

$$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

The cyclic arrangement of letters on the right-hand side of the equation should be carefully noticed, it will then be an easy matter to write down the corresponding formulae for $\cos B$ and $\cos C$ Thus,

$$\cos B=\frac{c^2+a^2-b^2}{2ca}, \quad \cos C=\frac{a^2+b^2-c^2}{2ab}$$

The preceding formulae, except in the case of comparatively simple numbers, involve somewhat tedious and troublesome calculations; hence, other formulae better adapted for the application of logarithms are generally used

Sine rule.—In a triangle ABC, the sines of the angles are proportional to the lengths of the opposite sides

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

From B, (Fig. 39) draw a line perpendicular to and meeting the side AC, in D. Then

$$\sin A=\frac{BD}{AB}=\frac{BD}{c},$$

$$\sin C=\frac{BD}{BC}=\frac{BD}{a}.$$

Hence, $$\frac{\sin A}{\sin C}=\frac{a}{c},$$

or $$\frac{\sin A}{a}=\frac{\sin C}{c}$$

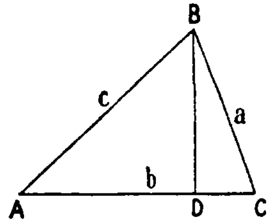

FIG 39

In a similar manner, if a line is drawn from C perpendicular to AB, we can prove

$$\frac{\sin A}{a}=\frac{\sin B}{b}.$$

Hence, $$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

The result shows that the greatest side subtends the greatest angle, and conversely

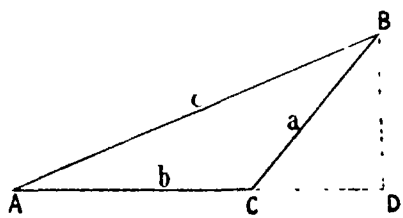

FIG 40

The results are also true when the given triangle is obtuse (Fig. 40)

Thus, $$\frac{BD}{c}=\sin A, \text{ or } BD=c\sin A.$$

Also, $$\frac{BD}{a}=\sin(180^\circ-C)=\sin C;$$

$$\therefore\ BD=a\sin C,$$

giving $$a\sin C=c\sin A;$$

$$\therefore\ \frac{a}{c}=\frac{\sin A}{\sin C} \text{ as before.}$$

Ex. 1. In a triangle ABC, given $A=38°$, $B=72°$, $c=2''66$ (Fig. 41), find the remaining sides of the triangle.

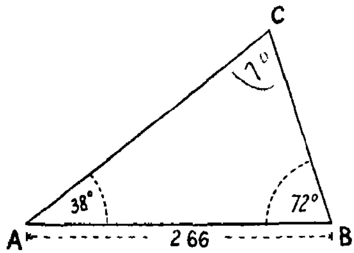

FIG 41.

Here

$$C=180°-(38°+72°)=70°,$$

$$\frac{b}{c}=\frac{\sin B}{\sin C};$$

$$b=\frac{c\sin B}{\sin C}$$

$$=\frac{2\cdot66\times\sin 72°}{\sin 70°}$$

$$=\frac{2\cdot66\times0\cdot9511}{0\cdot9397}.$$

$$\log b=\log 2\cdot66+\log 0\cdot9511-\log 0\cdot9397$$

$$=0\cdot4249+\bar{1}\cdot9782-\bar{1}\cdot9730=0\cdot4301-\log 2\cdot693,$$

$$b=2\cdot693.$$

Similarly, $$a=\frac{c\sin A}{\sin C}=\frac{2\cdot66\sin 38°}{\sin 70°}$$

$$=1\cdot743.$$

Ex. 2. At a certain place B, the angle of elevation of an object is 45°. At another place C, distant 200 ft. from B, and in a straight line with the object between them, the angle is 10° Find the distance from C to the object If the actual distance from B to C is 198 7 ft., and the angle at C is 10° 20′, what is the percentage difference in the answer?

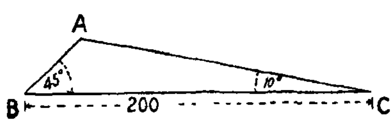

FIG 42

In Fig. 42 BC is 200 ft., and the angles at B and C are 45° and 10° respectively, A is the object, and the distance AC or b in the triangle ABC is required.

$$A=180°-(B+C)=180°-55°=125°$$

and $$\frac{b}{a}=\frac{\sin B}{\sin A}=\frac{\sin 45°}{\sin 125°};$$

$$\therefore b=\frac{200\sin 45°}{\sin 55°}=172\cdot6 \text{ ft}$$

When the angle at C is $10° 20'$, the angle at A

$$=(180-45°-10° 20')=124° 40',$$

and $\sin 124° 40' = \sin 55° 20'$;

$$b=\frac{198{\cdot}7 \times \sin 45°}{\sin 55° 20'}=171{\cdot}9 \text{ ft}$$

Hence, by comparison of lengths 172 6 and 171·9

$$\text{Difference}=\frac{0\ 7 \times 100}{170{\cdot}9}=0{\cdot}409\ \%\text{ in excess}$$

Ex. 3. In a triangle ABC, the base AB is 1000 feet long, and the angles at A and B are $31° 20'$ and $125° 19'$ respectively; find the length of the perpendicular let fall from C on AB produced, and the distance from A to the foot of the perpendicular

Let D (Fig. 43) be the foot of the perpendicular

$$b=\frac{1000 \sin 54° 41'}{\sin 23° 21'}$$

$$=\frac{1000 \times 0\ 8160}{0\ 3964}$$

$$=2059$$

$$CD=2059 \sin 31° 20'$$

$$=2059 \times 0\ 5200=1070$$

$$AD=2059 \sin 58° 40'$$

$$=2059 \times 0\ 8542$$

$$=1759.$$

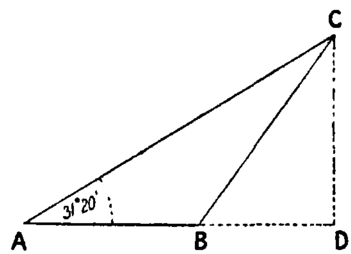

FIG 43

EXERCISES XVI

1. Two sides of a triangle are 2 5 and 3·75 respectively, the angle subtended by the longer side is 85°; find the remaining side and angles.

2 The angles at the base of a triangle measure 43° and 67° respectively; the base is 2″·5 long Find the remaining sides.

3. If $A=55°$, $B=65°$ and $c=270$, find a.

4 Given $b=105$, $c=55$, $A=51°$, find B and C.

5. In a triangle ABC, the base AB is 1000 feet long, and the angles at A and B are $31° 20'$ and $125° 19'$ respectively; find the length of the perpendicular let fall from B on AC, and the distance from B to the foot of the perpendicular.

6 Two angles of a triangle being 150° and $11° 40'$, and the longest side being 100 feet long; find the length of the shortest side.

7 In the triangle ABC, $A=60°\ 15'$, $B=54°\ 30'$ and $AB=100$ yards; find AC.

8 A station B is due north of a station A. Two cyclists leave A and B at the same time and ride along straight roads—AC, BC, to a station C, which bears 35° N. of E from A and 10° S of E from B Compare their average speeds if they reach C at the same time.

9 If the angles adjacent to the base of a triangle are $22°\cdot5$ and $112°\cdot5$, show that the perpendicular altitude will be one-half the base

10 A passenger on a steamer moving due north along a straight reach of a lake, at a uniform speed of 10 miles an hour, observed at a certain instant that the bearing of a tower on shore made an angle of 28° with the direction of the steamer, and 3 minutes later an angle of 54° Find the distance of the tower from the track of the steamer Find, also, the time from the second observation before the steamer will be abreast the tower.

11. In a triangle ABC, having a right angle at C, CB is 30 feet long and $BAC=20°$ CB is produced to a point P such that $PAC=55°$. What is the length of PC?

12. A bridge has five equal spans, each 100 feet in length A boat is moored in line with one of the two middle piers, and the whole length of the bridge subtends a right angle as seen from the bridge. Show that the distance of the boat from the bridge is 24.5 feet

Solution of a Triangle given its three sides—In any triangle ABC to prove that

$$a^2=b^2+c^2-2bc\cos A \quad \ldots \quad (1)$$

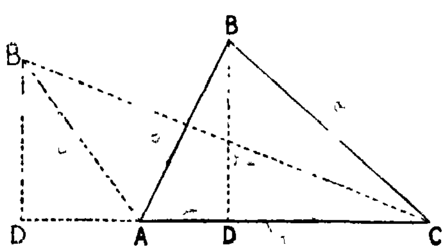

FIG. 44

From B (Fig. 44) draw BD perpendicular to the base AC and meeting it in D If the length AD be denoted by x, then $DC=b-x$.

Let $y=BD$. Then, from the right-angled triangle ABD,

$$AB^2=AD^2+DB^2,$$

or

$$c^2=x^2+y^2. \quad \ldots\ldots \quad \text{(ii)}$$

Similarly, from the right-angled triangle BDC,

$$a^2=(b-x)^2+y^2=b^2-2bx+x^2+y^2$$

Substituting from (ii),

$$a^2=b^2+c^2-2bx.$$

Also, $x=c\cos A$ because AD is the projection of AB on the base, $\therefore a^2=b^2+c^2-2bc\cos A.$

When the angle at A is an obtuse angle, then, with the same notation as before,

$$a^2=y^2+(b+x)^2=y^2+b^2+2bx+x^2.$$

Also, $c^2=x^2+y^2$

Substituting this value,

$$a^2=b^2+c^2+2bx$$

Also, $x=c\cos DAB$

But $\cos DAB=-\cos A$

Substituting, we obtain $a^2=b^2+c^2-2bc\cos A$

When the angle at A is $90°$ the triangle is right-angled.

But $\cos 90°=0$;

hence, $a^2=b^2+c^2$

Eq (i) may be written, $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

In a similar manner, if perpendiculars are let fall from A and C upon the opposite sides, the corresponding expressions for $\cos B$ and $\cos C$ may be obtained. Or, their values may be written down by noticing the cyclic arrangement of the letters. Thus,

$$\cos B=\frac{a^2+c^2-b^2}{2ac}, \text{ and } \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

From the three formulae for $\cos A$, $\cos B$, and $\cos C$, the angles of a triangle can be obtained when the three sides are given These expressions are chiefly useful for those cases where the given numbers are such that the operations

indicated can be readily carried out. When the numbers indicating the lengths of the sides consist of three or more figures, formulae adapted to logarithms should be used.

Ex. 1. The sides of a triangle are 5, 6 and 7 respectively. Find the three angles.

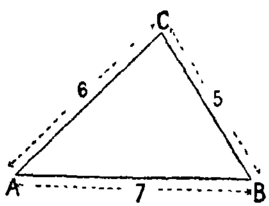

FIG 45

Using squared paper, set out AB as base and equal to 7 units (Fig 45). Then, from A and B as centres, with radii 6 and 5 units respectively, describe arcs intersecting in C. The angles can now be measured. Or, using the formula

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{6^2+7^2-5^2}{2\times 6\times 7} = \frac{60}{84}$$
$$=0{\cdot}7143.$$

From Table V, $A=44°\ 25'$.

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{5^2+7^2-6^2}{2\times 5\times 7} = \frac{38}{70} = \frac{19}{35} = 0{\cdot}5429;$$
$$\therefore B = 57°\ 7'.$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{5^2+6^2-7^2}{60} = \frac{1}{5} = 0{\cdot}2,$$
$$\therefore C = 78°\ 28'.$$

Ex. 2 Find the cosine of the largest angle of the triangle whose sides are 8 feet, 11 feet and 14 feet long respectively, and find the angle itself

Let the three sides 14, 11 and 8 be denoted by a, b and c respectively. The largest angle A is opposite the largest side a.

Then $$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{121+64-196}{2\times 11\times 8} = -\frac{1}{16} = -0{\cdot}0625$$

From Table V., $A=93°\ 35'$.

Formulae adapted to logarithmic computation.

To prove that $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

and $$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

where s denotes half the sum of the sides.

$$\cos A = 1 - 2\sin^2\frac{A}{2}, \ \dots \text{(p. 32)},$$

$$2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{2bc+a^2-b^2-c^2}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{(a-b+c)(a+b-c)}{2bc}.$$

But $\qquad s = \frac{1}{2}(a+b+c).$

Then $\qquad a-b+c = 2(s-b)$

and $\qquad a+b-c = 2(s-c)$

Hence, $\qquad \sin^2\frac{A}{2} = \frac{4(s-b)(s-c)}{4bc};$

$$\therefore \ \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}. \qquad \dots \dots \quad \text{(i)}$$

Again, $\qquad \cos A = 2\cos^2\frac{A}{2} - 1,$

$$\therefore \ 2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

$$\therefore \ \cos^2\frac{A}{2} = \frac{4s(s-a)}{4bc};$$

$$\therefore \ \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \qquad \dots \dots \quad \text{(ii)}$$

Dividing (i) by (ii), $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Also, $\qquad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2};$

$$\therefore \ \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Area of a triangle.—The area of a triangle can be found in any case when the triangle can be solved.

Let ABC (Fig. 46) be a triangle. The two sides, b and c, and angle A being known, the area of the triangle is $\frac{1}{2}p \times b$, where p is the length of the perpendicular BD.

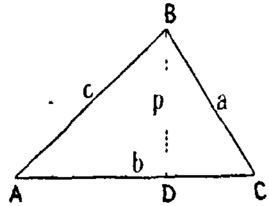

FIG. 46

Also, $p = c \sin A$. Hence,

$$\textbf{Area of triangle} = \tfrac{1}{2}\mathbf{bc \sin A}, \ldots \quad (1)$$

or one-half of the product of two sides and sine of included angle

When the included angle is a right angle or $90°$, then $\sin 90° = 1$, and Eq. (1) reduces to half the product of the sides which contain the right angle.

When the three sides of a triangle are given, it is only necessary to substitute in (1) the value of $\sin A$ (p 165).

$$\therefore \text{ area of triangle} = \tfrac{1}{2}bc \times \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

It is always advisable to check the results obtained from the above formulae by graphical methods

When only one angle is required, we may use the formula for $\sin\frac{A}{2}$, or $\cos\frac{A}{2}$; but if all the angles are required, the most suitable formula is

$$\tan\frac{\mathbf{A}}{\mathbf{2}} = \sqrt{\frac{(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{s}(\mathbf{s}-\mathbf{a})}},$$

because it will only be necessary to look out the logarithms for the four terms s, $(s-a)$, $(s-b)$ and $(s-c)$.

One method which may be used will be seen from the following example

Ex. 3. The sides a, b, c are 1·2, 1·6 and 2 feet respectively; find the angles of the triangle and its area

$$\left.\begin{array}{l} a = 1{\cdot}2, \\ b = 1{\cdot}6, \\ c = 2{\cdot}0, \end{array}\right\} \begin{array}{l} s \quad\;\; = 2{\cdot}4, \\ s-a = 1{\cdot}2, \\ s-b = 0{\cdot}8, \\ s-c = 0{\cdot}4. \end{array}$$

$$\tan\frac{A}{2}=\sqrt{\frac{0\cdot8\times0\cdot4}{2\cdot4\times1\cdot2}}=\sqrt{\frac{0\cdot32}{2\cdot88}}$$

$$\log\tan\frac{A}{2}=\frac{1}{2}(\log 0\cdot32-\log 2\cdot88)=\bar{1}\cdot5228\,;$$

$$\tan\frac{A}{2}=0\cdot3333.$$

From Table VI., $\dfrac{A}{2}=18^\circ\ 26'$;

$$A=36^\circ\ 52'$$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{1\cdot2\times0\cdot4}{2\cdot4\times0\cdot8}}$$

$$=\sqrt{\frac{0\cdot48}{1\cdot92}}=\sqrt{\frac{1}{4}}=0\cdot5\,;$$

$$\tan\frac{B}{2}=0\cdot5,$$

$$\frac{B}{2}=26^\circ\ 34'\,;$$

$$B=53^\circ\ 8'$$

Having found A and B, then C is known from the relation,

$$A+B+C=180^\circ,$$

$$C=180^\circ-(A+B)=90^\circ$$

$$\text{Area}=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{2\cdot4\times1\cdot2\times0\cdot8\times0\cdot4}$$

$$=0\cdot96\text{ square feet}$$

Ex. 4. The sides a, b, c of a triangle are 5, 6, and 7 inches respectively. Find the smallest angle.

The smallest angle will be the angle opposite the side a

$$s=\frac{1}{2}(5+6+7)=9,\quad s-b=3,\quad s-c=2\,,$$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}=\sqrt{\frac{3\times2}{6\times7}}=\sqrt{\frac{1}{7}}\,;$$

$$\frac{A}{2}=22^\circ\ 12',$$

$$A=44^\circ\ 24'$$

Ex. 5. The three sides of a triangle are 3745 ft., 5762 ft. and 7593 ft respectively Find the largest angle.

$$\tan^2 \frac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)} = \frac{4805 \times 2788}{8550 \times 957};$$

$$\frac{1}{2}A = 52^\circ \text{ nearly},$$

$$A = 104^\circ.$$

EXERCISES XVII

1. The three sides a, b, c of a triangle are $\sqrt{6}$, 2 and $\sqrt{3}+1$ respectively; find the angles A, B and C.

2 The sides of a triangle are 242, 1212 and 1450 yards respectively; show that the area is 6 acres.

3 The sides a, b, c of a triangle are 0·9, 1 2 and 1 5 respectively; find the angle B and the area of the triangle.

4 The sides of a triangle are as 4 5 . 6 , find the angle opposite to the side 5

5. The sides of a triangle are 35, 40 and 45 feet respectively; find the largest angle

6 The sides a, b, c of a triangle are 12 5, 12·3 and 6 2 respectively; find $\sin \frac{1}{2}B$ and also B

7 The sides of a triangle are 1 8, 1·2 and 1 ft respectively; find the angles

8. The sides of a triangle being 20 ft , 21 ft and 29 ft , find the angle subtended by the side 29, also find the area of the triangle Prove the formulae you use

9. Given $a=13$, $b=9$, $c=12$; find the numerical value of $\tan\frac{A}{2}$, and then the angle A

10 The sides of a triangle are 5·25 feet, 6 50 feet and 7 77 feet respectively; determine the smallest angle

11. Find the smallest angle of the triangle whose three sides are 200, 250, 300 feet respectively.

12 Find the smallest angle of the triangle whose sides are 8, 9 and 13 units respectively

13. In a triangle ABC, $a=17$, $b=20$, $c=27$, find $\tan\frac{1}{2}A$; also find A

14 Determine the smallest angle in the triangle whose sides are in the ratio of 9 10 . 11.

15. Determine the smallest angle and the area of the triangle whose sides are 72·7 ft., 129 ft and 113 7 ft. Prove any formula you may use in the calculation

16 Prove the formula $\tan\frac{A}{2} = \left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}^{\frac{1}{2}}$, and use it to find the angles of a triangle whose sides are 4002, 9760 and 7942 feet respectively

17 In a triangle ABC, $a=\sqrt{5}$, $b=2$, $c=\sqrt{3}$; show that $8 \cos A \cos C = 3 \cos B$

18 The sides of a triangle are 36, 48 and 60 feet respectively; find the values of the angles opposite to them

19 In a triangle ABC, given $a=3$, $b=2\ 75$, $c=1\ 75$ ft, find the angle B; also find the length of the side of a square the area of which is equal to the given triangle.

20 The sides of a triangle are 1 3 ft, 1 4 ft and 1·5 ft; a rectangle equal in area to the given triangle has one side 1·4 ft. long; find the remaining side.

21 The diagonals of a parallelogram make an angle of 35° with one another, and are severally 117 72 and 157 41 feet long What is the area of the parallelogram?

22 (*a*) Find a formula for the area of a rectangle, having given a diagonal and an angle contained by the diagonals (*b*) If the diagonal is 63 86 ft. long, and the angle 106 9′, calculate the area

23. Find a formula for the area of a parallelogram, having given the diagonals and the angle between them If the diagonals are 30 ft and 55·44 feet long, and the angle 146° 54′, calculate the area.

24 If the sides of a triangle be 7 152 inches, 8 263 inches, 9 375 inches, find its area

25 The sides of a triangle are 1·3, 1·4 and 1·5 feet respectively; find the angles.

Show that the area of this triangle is 0·84 square feet. What is the area of a triangle whose sides are 13, 14 and 15 feet respectively?

26. The three sides of a triangle are 524, 566 and 938 feet respectively Determine the three angles

Solution of a triangle given two sides and the included angle.—When the data include two sides and included angle, we may use the formula

$$\tan\tfrac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2},$$

which may be obtained as follows

From the sine rule (p. 159)

$$\frac{\sin B}{\sin C}=\frac{b}{c};$$

$$\therefore \frac{\sin B-\sin C}{\sin B+\sin C}=\frac{b-c}{b+c}.$$

Using the results given (p 28), we obtain

$$\frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\cos\frac{B-C}{2}\sin\frac{B+C}{2}}=\frac{b-c}{b+c};$$

$$\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{b-c}{b+c},$$

or $\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}, \ldots \left(\text{since } \frac{B+C}{2}=90-\frac{A}{2}\right).$

This determines $(B-C)$, and since $B+C=180-A$, it follows that B and C can be obtained

Ex 1. The sides b and c of a triangle are 5·35 ft and 4 65 ft, the angle between the two given sides is 51° 20′ Find the remaining angles

Here $A=51^\circ\ 20'$; $\frac{A}{2}=25^\circ\ 40'$

$$\cot\frac{A}{2}=\cot 25^\circ\ 40'=\tan(90^\circ-25^\circ\ 40')=\tan 64^\circ\ 20';$$

$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\tan 64^\circ\ 20'$$

$$=\frac{5{\cdot}35-4\ 65}{5{\cdot}35+4\ 65}\times 2\ 081=0\ 07\times 2\ 081,$$

or $\tan\frac{1}{2}(B-C)=0\ 14567;$

$\therefore \frac{1}{2}(B-C)=8^\circ\ 17'.$

Also, $\frac{1}{2}(B+C)=64^\circ\ 20';$

$\therefore$ by addition, $B=72^\circ\ 37'.$

By subtraction, $C=56^\circ\ 3'.$

Ex. 2. Given the two sides of a triangle b and c equal to 3·45 and 1·74 ft respectively, and angle $A=37°\ 20'$; find the angles B and C, the remaining side a, and the area of the triangle.

$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$=\frac{3\cdot45-1\cdot74}{3\cdot45+1\cdot74}\cot 18°\ 40'$$

$$=\frac{1\cdot71}{5\cdot19}\times2\cdot9600=0\cdot9752.$$

$$\tfrac{1}{2}(B-C)=44°\ 17',$$

$$\tfrac{1}{2}(B+C)=71°\ 20',\ \text{i.e.}\ \tfrac{1}{2}(180°-37°\ 20'),$$

$$B=115°\ 37',\ \text{and}\ C=27°\ 3'$$

The side a may be found from the relation

$$\frac{a}{c}=\frac{\sin A}{\sin C};$$

$$a=\frac{1\cdot74\sin 37°\ 20'}{\sin 27°\ 3'}=\frac{1\cdot74\times0\cdot6065}{0\cdot4548}=2\cdot32\text{ ft}$$

$$A=\text{area of triangle}=\tfrac{1}{2}bc\sin A$$

$$=\tfrac{1}{2}\times3\cdot45\times1\cdot74\sin 37°\ 20'$$

$$=1\cdot725\times1\cdot74\times0\cdot6065,$$

$$\therefore\ \log(A)=\log1\cdot725+\log1\cdot74+\log0\cdot6065=0\cdot2601;$$

$$A=1\cdot82\text{ sq. ft}$$

Ex. 3. Two sides of a triangle are measured and found to be 32·5 and 24·2 inches, the included angle being 57°; find the area of the triangle. If the true lengths of the sides are 32·6 and 24·1, what is the percentage error in the answer?

$$\text{Area}=\tfrac{1}{2}\times32\cdot5\times24\cdot2\times\sin57°=329\cdot8\text{ sq. in}$$

$$\text{,,}\quad=\tfrac{1}{2}\times32\cdot6\times24\cdot1\times\sin57°=329\cdot5\quad\text{,,}$$

$$\text{Error}=0\cdot3\text{ sq in.; percentage error}=\frac{0\cdot3\times100}{329\cdot5}$$

$$=0\cdot09\,\%.$$

Ex. 4 Two sides of a triangle are 385 feet and 231 feet respectively, and the included angle is 50°.

Find the other two angles and the remaining side

$$\tan \tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\tan \tfrac{1}{2}(B+C)$$

$$=\frac{385-231}{385+231}\tan 65^\circ$$

$$=\frac{154}{616}\times 2\cdot 1445=0\cdot 5361.$$

From Table VI,

$$\tfrac{1}{2}(B-C)=28^\circ\ 12'$$

Also,

$$\tfrac{1}{2}(B+C)=65^\circ,$$

$$\therefore B=93^\circ\ 12',\quad C=36^\circ\ 48'$$

Ex. 5 ABC is a triangle in which a and b are together twice c; show that the area equals $3c^2\tan\frac{1}{2}C\div 4$.

What is the greatest value of C consistent with the given conditions?

$$a+b=2c;$$

$$s=\frac{1}{2}(a+b+c)=\frac{3c}{2}.$$

Let A denote the area of the triangle.

$$A=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=s(s-c)\sqrt{\frac{(s-b)(s-c)}{s(s-c)}},\text{ but }\sqrt{\frac{(s-b)(s-a)}{s(s-c)}}=\tan\tfrac{1}{2}C$$

$$=\frac{3c}{2}\times\frac{c}{2}\tan\tfrac{1}{2}C$$

$$=3c^2\tan\tfrac{1}{2}C\div 4$$

If $a+b=2c$, $\quad \sin A+\sin B=2\sin C,$

$$2\sin\frac{A+B}{2}\cos\frac{A-B}{2}=4\sin\frac{C}{2}\cos\frac{C}{2}\quad\ldots\quad (1)$$

But since

$$A+B+C=180,$$

$$\sin\frac{A+B}{2}=\cos\frac{C}{2},$$

$$\therefore \sin\frac{C}{2}=\frac{1}{2}\cos\frac{A-B}{2}\text{ from (1).}$$

Obviously, $\frac{C}{2}$ is greatest when $A=B$

In which case $\sin\frac{C}{2}=\frac{1}{2}=\sin 30^\circ$;

$\therefore C=60^\circ$.

EXERCISES. XVIII.

1 The sides of a triangle are 535 feet and 465 feet, and the angle between them is $51^\circ\ 29'$; find the other angles to the nearest minute.

2 In a triangle ABC, given $b=400$ feet, $c=100$ feet and $A=64^\circ\ 20'$. find B and C

3 In a triangle ABC. given $a=3$, $b=5$ and $C=120^\circ$; find $\tan\frac{1}{2}(B-A)$

4 In a triangle $A=60^\circ$, $b=9$, $c=6$; find the other angles.

5 In a triangle ABC, $b=14$, $c=11$, $A=60^\circ$; find the other angles

6 In a triangle ABC, $\frac{b}{c}=\frac{3}{7}$ and $A=6^\circ\ 37'$, find the other angles.

7 Two of the sides of a triangle are 11 and 5 respectively, and the included angle is 60°; find the other angles. Also find the lengths of the other side of the triangle.

8 Two sides of a triangle are 1·5 and 13·5 respectively, and the included angle is 65°; find the remaining angles.

9 Two sides of a triangle are 4 feet and 6 feet in length respectively, and the included angle is 30°, find the area of the triangle

10. Two sides of a triangle are 9 and 7 feet respectively, and the angle between them is 60°; find the other angles.

11. The two sides AB and BC of a triangle are 44·7 and 69·8 respectively, the angle ABC is 32°. (*a*) Find the length of the perpendicular from A to BC, (*b*) the area of the triangle ABC, (*c*) the angles A and C

12 Two sides of a triangle are 729 and 353 feet respectively, and the included angle is 54°, find the other angles, and the remaining side.

13. Two sides of a triangle are 3747 and 1528 feet respectively, the included angle is 33°; find the other two angles.

14. Prove that the area of a triangle $ABC=\frac{a^2\sin B\sin C}{2\sin A}$. If $A=75^\circ$, $C=60^\circ$ and $a=2(1+\sqrt{3})$, show that the area is equal to $6+2\sqrt{3}$.

Solution of a triangle given two of its sides and the angle opposite one of these sides.—When the given data include two sides and the angle opposite one of these, we may use the sine rule. Thus, let a and b (Fig. 47) be the two sides and A the given angle.

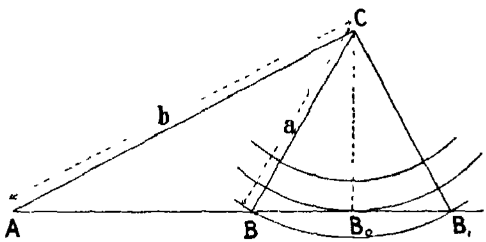

FIG 47

The angle B may be obtained from the relation

$$\sin B = \frac{b}{a} \sin A.$$

The angle B is obtained from its sine, but, as two angles less than 180° may have the same sine, this case is usually known as the **ambiguous case**

This may be shown graphically as follows

Draw two lines, AC and AD, at an angle A (Fig 47). Along one side measure a length $AC = b$. From C as centre and radius a, describe an arc of a circle.

(i) If this cuts the base AD in two points B and B_1, then, on joining B and B_1 to C, we obtain two triangles ABC or AB_1C, either of which satisfies the given conditions

But if a is greater than b there is only one triangle

(ii) If the circle touches AD at B_0 then the triangle is right-angled.

(iii) If the circle does not cut AD (as indicated), then there is no solution.

It will be seen that the three conditions just referred to are obtainable from Eq. (i) as follows

(i) Thus, if $b \sin A < a$, $\sin B$ is < 1, and there may be two solutions.

(ii) When $b \sin A = a$, then $\frac{b \sin A}{a} = 1$;

$$\sin B = 1 \text{ and } B = 90^\circ.$$

Hence, the triangle is a right-angled triangle

(iii) If $b \sin A > a$, then $\frac{b \sin A}{a}$ is greater than unity, and there is no triangle with the given parts

Algebraic solution —It will be obvious from the preceding paragraph that from the data given we may obtain two values, one value, or an imaginary or impossible value of the remaining side c

Thus, from the equation

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$c^2 - 2bc \cos A = a^2 - b^2.$$

This is a quadratic equation from which to find c,

$$c^2 - 2bc \cos A + (b \cos A)^2 = a^2 - b^2 + b^2 \cos^2 A$$

$$= a^2 - b^2(1 - \cos^2 A)$$

$$= a^2 - b^2 \sin^2 A,$$

$$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}$$

(i) If $b \sin A < a$, there are two values of c

(ii) If $b \sin A = a$, the two roots are equal

(iii) If $b \sin A > a$, the quantity under the root sign is negative, and the values, or roots, are imaginary

EXERCISES. XIX.

1 Find the angle A of the triangle ABC, having given that $AC = 257$ feet, $BC = 650$ feet and $C = 90^\circ$. Find also the length of the line AD which meets BC in D, so that the angle ADC is $40^\circ\ 32'$.

2 Find the value or values of c, having given $A = 35^\circ\ 36'$, $a = 1770$, $b = 2164$

3 Find all the parts of the triangles which have one side 90 feet long, another side 60 feet long, and the angle opposite to the shorter side equal to $18^\circ\ 37'$.

4. Given $b = 8.4$ inches, $c = 12$ inches, $B = 37^\circ\ 36'$; find A

5. In any triangle, if $A = 47^\circ$, $a = 180$, $b = 215$; find B.

6. In a triangle ABC, given $AC = 166\cdot5$ feet, $BC = 162\cdot5$ feet, the angle $A = 52^\circ\ 19'$ Solve either of the triangles to which the data belong.

7. Given $A = 40^\circ$, $a = 140\cdot5$, $b = 170\cdot6$; find B

8. In the triangle ABC, $A = 26^\circ\ 26'$, $b = 127$ and $a = 85$; find B

9 Two angles of a triangular field are $22\frac{1}{2}^\circ$ and 45° respectively, and the length of the side opposite to the latter is a furlong. Show that the field contains exactly two acres and a half.

10 The lengths of two sides of a triangle are $537\cdot4$ feet and $158\cdot7$ feet, the angle opposite the shorter side is $15^\circ\ 11'$ Calculate the other angles of the triangle, or of the triangles, if there are two

11 Having given $A = 30^\circ$. $a = \sqrt{2}$, $c = 2$, solve the triangle

12 In a given triangle $a = 145$, $b = 178$. $B = 41^\circ\ 10'$, find A.

13. Given $B = 30^\circ$, $c = 150$, $b = 50\sqrt{3}$; show that of the two triangles that satisfy the data, one will be isosceles and the other right-angled. (i) Find the third side in the greatest of these triangles; (ii) would the solution be ambiguous if the data had been $B = 30^\circ$, $c = 150$, $b = 75$?

Measurement of heights and distances.—The angle made with the horizontal plane by a straight line joining a point of observation to a distant point, when the point is above the point of observation, is called the **angle of elevation** The angle is called the **angle of depression** when the distant point is below the horizontal line through the point of observation These angles are measured by an instrument called a **Theodolite**

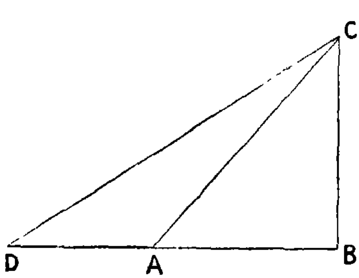

Fig 48

The angle subtended by a line joining two distant objects may be measured by a **Sextant**

Thus, if A (Fig 48) denotes the place of observation, and C a distant point above A, then the angle, between the line joining A to C and a horizontal line AB, is the angle of elevation of C.

If CB be drawn perpendicular to AB and meeting AB in B (Fig. 48), then the height of the object B can be obtained when AB and the angle at A are given.

Since $$\frac{BC}{AB} = \tan A\,;$$

$$BC = AB \tan A. \quad . \quad . \quad . \quad \quad . \quad . \quad . \quad (1)$$

The plan adopted is to write the fraction so that the unknown quantity is the numerator and the known quantity the denominator

When it is either impossible or inconvenient to obtain the distance AB, a distance such as DA in the line BA produced (Fig 48) may be measured and the angles of elevation ADC and BAC obtained. Denoting the known length DA by l, and the distance AB by x, then if h denotes the height BC, $h = x \tan BAC$ (i)

Also, $$h = (l + x) \tan BDC \ldots \ldots \ldots \quad \text{(ii)}$$

If we substitute the value of h from (i) in (ii), we obtain a simple equation in x, and finally h may be found from (i).

Angles of depression.—If a horizontal line be drawn through C, then the angles at C subtended by two objects D and A, are called angles of depression, and the solution is effected as in the preceding case

Ex 1 At a distance of 99 ft from the foot of a tower the angular elevation is 60° Find the height of the tower.

If h denote the height, then

$$h = 99 \tan 60° = 99 \times \sqrt{3}$$
$$\log h = \log 99 + \tfrac{1}{2} \log 3 = 2\ 2341\,;$$
$$h = 171\ 4 \text{ ft}$$

This result may be verified by construction, as in Fig 48 Draw a right-angled triangle having the angle at $A = 60°$ and $AB = 99$ Then $BC = 171\ 4$

Ex. 2 The elevation of an object on a hill is observed, from a certain place in the horizontal plane through its base, to be 15°. After walking 120 feet towards it on level ground the elevation is found to be 18° Find the height of the object and its distance from the second place of observation

Draw a line DAB and from D set off DA (Fig. 49) to represent 120 feet, and make the angles BAC and BDC equal to $18°$ and $15°$ respectively From C, the point of intersection, draw BC perpendicular to DA and meeting DA produced in B Then BC is the height and BA is the distance required

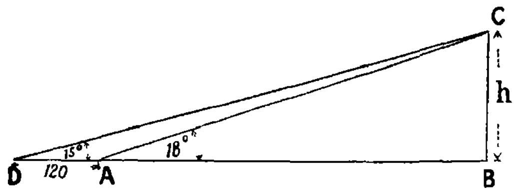

FIG. 49

Let $BA=x$ and $BC=h$ By calculation, two or more methods may be used to find x and h If necessary, one method may be used to check another

First method As the angle $BAC=ADC+ACD$, the angle $ACD=3°$;

$$\therefore \frac{AC}{AD}=\frac{\sin 15°}{\sin 3°},$$

or $$AC=\frac{AD \sin 15°}{\sin 3°}.$$

Again $$BC=AC \sin 18° ;$$

$$\therefore h=\frac{AD \sin 15° \sin 18°}{\sin 3°}=\frac{120\times 0\,2588\times 0\,3090}{0\,0523}$$

$$=183\,5 \text{ ft}$$

Also, $$x=h \cot 18°$$

$$=183\,5\times 3\,0777=564\,76 \text{ ft.}$$

Second method. Using the same notation,

$$h=x \tan 18°. \qquad \ldots \qquad (i)$$

Also, $$h=(120+x)\tan 15°. \qquad \ldots \qquad (ii)$$

Substitute in (ii) the value of h from (i);

$$\therefore x \tan 18°=120 \tan 15°+x \tan 15°,$$

or $$x(\tan 18°-\tan 15°)=120 \tan 15° ;$$

$$\therefore x=\frac{120 \tan 15°}{\tan 18°-\tan 15°}=\frac{120\times 0\,2679}{0{\cdot}057},$$

$$\therefore x=564\,76 \text{ ft.}$$

Substituting this value for x in (1), h is obtained.

In the preceding example the angle of elevation has been used. A similar method is employed when the angles of depression are given.

Ex. 3. From the top of a hill, the angles of depression of two objects on a horizontal plane through the base of a hill are found to be 15° and 18° respectively Find the height of the hill, the distance between the objects being 120 feet

Draw a horizontal line passing through C (Fig 49) Make the angles of depression equal to 15° and 18° respectively Draw a horizontal line DA equal to 120 ft Produce DA to meet a line CB perpendicular to DA in B Then BC is the height required

As a good exercise in manipulation of symbols it is interesting to solve the preceding question, assuming that the data consist of letters instead of numerical quantities.

Let the angles BAC and BDC be denoted by α and β respectively, the distance AD by l, the remaining quantities as in the preceding

Then
$$\frac{DC}{l} = \frac{\sin DAC}{\sin DCA}$$

$$= \frac{\sin(180° - \alpha)}{\sin(\alpha - \beta)} = \frac{\sin\alpha}{\sin(\alpha - \beta)};$$

$$DC = \frac{l\sin\alpha}{\sin(\alpha - \beta)},$$

$$h = DC\sin\beta = \frac{l\sin\alpha\sin\beta}{\sin(\alpha - \beta)},$$

and substituting numerical values for l, α and β it will be seen that the result agrees with the preceding result

Ex. 4 From a station h feet above the water the angular depression from the horizontal of the light of a passing vessel and of its reflection in the water was observed to be D_1 and D_2 minutes; prove that the horizontal distance of the vessel was

$$2h \operatorname{cosec}(D_1 + D_2)\cos D_1 \cos D_2 \text{ feet}$$

If the angle D_1 and D_2 are small, prove that the distance is practically

$$\frac{3438h}{\frac{1}{2}(D_1 + D_2)} \text{ feet, or } \frac{1146h}{\frac{1}{2}(D_1 + D_2)} \text{ yards.}$$

Let P denote the station at a distance h feet above the surface of the water AL (Fig. 50), the angle $MPL = D_1$ and $MPK = D_2$.

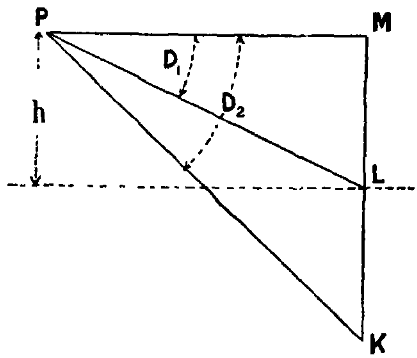

FIG 50.

Let S denote the horizontal distance PM where M is vertically over L

Then $ML = S \tan D_1$,

$MK = S \tan D_2$;

$\therefore \quad ML + MK = S(\tan D_1 + \tan D_2)$;

$$2h = S(\tan D_1 + \tan D_2) = S\left(\frac{\sin D_1}{\cos D_1} + \frac{\sin D_2}{\cos D_2}\right)$$

$$= S\left(\frac{\sin D_1 \cos D_2 + \sin D_2 \cos D_1}{\cos D_1 \cos D_2}\right)$$

$$= \frac{S \sin(D_1 + D_2)}{\cos D_1 \cos D_2},$$

$$\therefore \quad S = \frac{2h \cos D_1 \cos D_2}{\sin(D_1 + D_2)} \qquad \text{(i)}$$

$$= 2h \operatorname{cosec}(D_1 + D_2) \cos D_1 \cos D_2$$

When D_1 and D_2 are small angles, $\cos D_1$ and $\cos D_2$ may each be taken to be unity

Hence, from (i), $$S = \frac{2h}{\sin(D_1 + D_2)} \quad . \quad . \qquad \text{(ii)}$$

Also, when an angle is small the sine of an angle is approximately equal to the radian measure of the angle, substituting in (ii);

$$\therefore \quad S = \frac{2h}{\frac{3 \cdot 1416}{180 \times 60}(D_1 + D_2)} = \frac{3438h}{\frac{1}{2}(D_1 + D_2)} \text{ ft} = \frac{1146h}{\frac{1}{2}(D_1 + D_2)} \text{ yds.}$$

When in problems concerned with heights and distances the data include the points of the compass, it is desirable to draw a perspective view; for even if such a sketch is only a rough approximation, it tends to clearness

Ex. 5. The angle of elevation of a steeple at a place due south of it is 45°, and at another place due west of the former the angle is 16°. If the distance between the two places is 100 feet, find the height of the steeple.

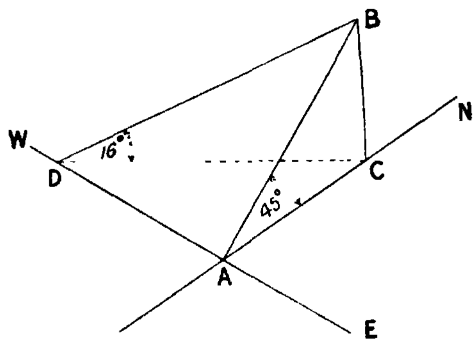

FIG. 51

Let BC (Fig. 51) denote the steeple, A the first place and D the second place of observation

$$BC = h = CD \tan 16^\circ, \quad \text{or} \quad h^2 = (CD)^2 \tan^2 16^\circ \qquad (1)$$

Also, as BAC is 45°, AC is equal to h.

$$CD^2 = 100^2 + h^2.$$

Substituting this value in (1),

$$h^2 = (100^2 + h^2)\tan^2 16^\circ = 100^2 \tan^2 16^\circ + h^2 \tan^2 16^\circ;$$

$$h^2(1 - \tan^2 16^\circ) = 100^2 \tan^2 16^\circ,$$

$$h^2 = \frac{100^2 \times 0{\cdot}2867^2}{1 - 0{\cdot}2867^2} = \frac{100^2 \times 0{\cdot}2867^2}{1{\cdot}2867 \times 0{\cdot}7133}$$

$$2 \log h = 2(\log 100 + \log 0{\cdot}2867) - \log 1{\cdot}2867 - \log 0{\cdot}7133,$$

$$\text{or} \quad \log h = 1{\cdot}4760 = \log 29{\cdot}92;$$

$$\therefore \quad h = 29{\cdot}92 \text{ feet}.$$

EXERCISES XX.

1. A person standing on one bank of a river observes the altitude of the top of a tower on the edge of the opposite side to be 55°; after receding 30 feet, he finds it to be 48°. Determine the breadth of the river.

2. Calculate the height of a tower from the following data: angles 20° and 55°; distance between points of observation 1000 feet in a direct line from the foot of the tower.

3. AB is a horizontal line 1300 ft. long. A vertical line is drawn from B upwards, and in it two points P and Q are taken, such that BQ is three times BP; BAP is 10° 30′. Calculate BP and BAQ.

4. The summit of a spire is vertically over the middle point of a horizontal square enclosure, whose side is a ft long, the height of the spire is h ft above the level of the square. If the shadow of the spire just reaches a corner of the square when the sun has an altitude θ, show that

$$h\sqrt{2} = a \tan \theta$$

Calculate h, having given $a = 1000$ ft., $\theta = 27°\ 29'$.

5 AB is a line 2000 feet long, B is due east of A; at B a distant point P bears 46° west of north, at A it bears 8° 45′ east of north; find the distance from A to P

6 The angle of elevation of a tower at a distance of 20 yards from its foot is three times as great as the angle of elevation 100 yds. from the same point Show that the height of the tower is $\frac{300}{\sqrt{7}}$ ft.

7 (a) The angular elevation of a tower from a certain station is A; at another station, in the same horizontal plane, and a feet nearer the tower, the angular elevation is $(90° - A)$, if h be the height of the tower above the horizontal plane, show that

$$h(1 - \tan^2 A) = a \tan A$$

(b) Calculate h, when $A = 30°$ and $a = 100$ feet.

8. ABC is a triangle in a horizontal plane, with a right angle at C, and P is the middle point of AB; a flagstaff is set up at C, and it is found that its angles of vertical elevation at A, B and P are α, β, θ, show that $\tan^2\theta = 2 \tan\alpha \tan\beta \sin 2A$

9 The foot, C, of a tower and two stations, A and B, are in the same horizontal plane The angular elevation of the tower at A is 60° and at B it is 45°, the distance from A to B is 100 feet and the angle ACB is 60°; show that the height of the tower is approximately 115 feet

10 P and Q are two stations 1000 yards apart on a straight stretch of sea shore, which bears East and West At P a rock bears 42° West of South, at Q it bears 35° East of South. Show that the distance of the rock from the shore is

$$1000 \sin 48° \sin 55° - \sin 77° \text{ yards},$$

and calculate this distance to the nearest yard

11 Find the length of an arc on the sea which subtends an angle of one minute at the centre of the earth, supposing the earth a sphere of diameter 7920 miles Give the answer in miles to three places of decimals

12 A person standing due south of a lighthouse observes that his shadow, cast by the light at the top, is 24 feet long; on walking 100 yards due east he finds his shadow to be 30 feet. Supposing that he is 6 feet high, find the height of the light from the ground.

13. The angle of elevation of a cliff at a certain place is 12° 30′, and at a second place of observation, distant 950 yards from the first and in a direct line towards the base, the second altitude is found to be 69° 30′ Find the height of the cliff

14. A tower stands at the foot of a hill whose inclination to the horizon is 9°, at a point 100 feet up the hill the tower subtends an angle of 54°; find its height.

15 The angles of elevation of a tower from the two ends of a measured line in the same horizontal plane as the base of the tower are 30° and 18° respectively Find the height of the tower in terms of l, the length of the measured line

16. The angle of elevation of a balloon from a station due south of it is 47° 20′, and from another station due west of the former on the same horizontal plane, and distant 671 3 feet from it, the elevation is 41° 15′ Find the height of the balloon

17 The angular elevation of a steeple at a place due south of it is 45°, and at another place due west of the former station and 100 yards from it the elevation is 15°. Find the height of the steeple

18. From the top of a tower, whose height is 100 feet, the angles of depression of two small objects on the plain below, which are in the same vertical plane with the tower, are observed, and found to be 45° and 30°; find to one decimal place the distance between them.

19. A person observes that two objects A and B bear due N. and 30° W of N, respectively. On walking a mile in the direction N.W., he finds that the bearings of A and B are N E. and due E. respectively, find the distance between A and B.

20. The altitude of a certain rock is observed to be 47°, and after walking 1000 feet towards the rock, up a slope inclined at an angle of 32° to the horizon, the observer finds that the altitude is 77°. Find the vertical height of the rock above the first point of observation.

21. From two stations A and B on shore, 3742 yards apart, a ship C is observed at sea. The angles BAC, ABC are simultaneously observed to be 73° and 82°, respectively. Find the distance from A to the ship.

22. A tower, whose height is known to be 100 feet, stands on a vertical cliff; the angle subtended by the tower at the eye of an observer in a boat at sea level is found to be 28°, and at the same station the cliff subtends an angle of 31°. Find the height of the cliff above sea level and the distance of the boat from the foot of the cliff.

23. ABC is a triangle in a horizontal plane, and D is a point vertically above C; if $AB = 600$ feet, $ACB = 117° \ 16'$, $CAD = 28° \ 28'$, and $CBD = 13° \ 32'$, show that

$$\tan \tfrac{1}{2}(BAC - ABC) = \sin 14° \ 56' \tan 31° \ 22' \div \sin 42°,$$

and calculate the length of CD.

24. A man standing due south of a spire finds the angular elevation of its summit to be α. He then walks to a point a yards due west of his former position and finds the elevation to be β. Show that the height of the spire in yards is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{\sin(\alpha - \beta) \sin(\alpha + \beta)}}.$$

25. A flagstaff is fixed on the top of a tower standing on a horizontal plane. An observer finds that the angles subtended at a point in the horizontal plane by the tower and the flagstaff are respectively α and β. He then walks a distance c directly towards the tower, and finds that the flagstaff subtends the same angle β as before. Prove that the heights of the tower and the flagstaff are respectively

$$\frac{c \sin \alpha \cos(\alpha + \beta)}{\cos(2\alpha + \beta)} \text{ and } \frac{c \sin \beta}{\cos(2\alpha + \beta)}.$$

26. A flagstaff a feet high is on a tower $3a$ feet high; prove that, if the observer's eye is on a level with the top of the staff and the staff and tower subtend equal angles, the observer is at a distance $a\sqrt{2}$ from the top of the staff.

27. The plane of a rectangular target is vertical and lies east and west; compare the area of the shadow on the ground with the area of the target when the sun is 10° from the south at an altitude of 64°.

CHAPTER IX.

AREA

Area.—The reader has already studied the areas and volumes of simple solids in an elementary course, and it is therefore only necessary here to collect the results for reference.

Parallelogram.—The area of a parallelogram is the product of the number of units of length in the base AB (Fig 52) and in the width BC

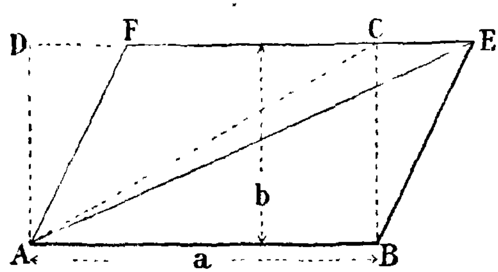

FIG. 52.—Area of a parallelogram

If a denotes the length of the base AB and b the width or height of BC, then

$$\mathbf{A = a \times b},$$

where A denotes the area of the parallelogram

The **rectangle**, as shown by the dotted lines (Fig 52), is a particular case of the parallelogram in which all the angles are right angles When, in addition, the four sides of a rectangle are equal, the four-sided figure is called a **square**, and $A = a^2$. The area is also one-half the product of the two diagonals by the sine of the angle of inclination.

Rhombus.—When the four sides of a parallelogram are equal, but the angles are *not* right angles, the figure is called a rhombus, and its **area is one-half the product of the two diagonals**

Triangle—Any parallelogram is divided into two equal parts by a diagonal (Fig. 52). Hence, when the base and height of a triangle are given, the **area of a triangle is one-half the product of the base and the height** As any side may be considered as the base of a triangle, the rule may be stated thus the **area of a triangle is equal to one-half the product of any side of a triangle and the length of the perpendicular let fall on that side from the opposite angle**

If p denote the length of the perpendicular BD (Fig 46, p 166),

$$\text{area} = \tfrac{1}{2} \times bp,$$

but $p = c \sin A$,

$$\therefore \text{ area of triangle} = \tfrac{1}{2}bc \sin A, \quad \ldots \quad \text{(i)}$$

or **area of a triangle is one-half the product of two sides and the sine of the included angle** The equivalent formulae for the remaining angles B and C are similarly

$$\tfrac{1}{2}ac \sin B \text{ and } \tfrac{1}{2}ab \sin C.$$

Area of a triangle in terms of the three sides.—Referring to p. 167,

$$\text{area of triangle} = \sqrt{s(s-a)(s-b)(s-c)},$$

where s denotes one-half the sum of the three sides

$$= \tfrac{1}{2}(a+b+c)$$

Length of perpendicular.—The formulae above may be used to obtain the length of the perpendicular from any angle on to the opposite side

Ex 1. The sides of a triangle are 5, 6 and 7 inches respectively Find the length of the perpendicular on the shortest side from the opposite angle

If p denote the length, then

$$\text{area of triangle} = \tfrac{1}{2}p \times 5 = \sqrt{s(s-a)(s-b)(s-c)},$$

where $$s = \tfrac{1}{2}(5+6+7) = 9;$$

$$\therefore p = \frac{2\sqrt{9 \times 4 \times 3 \times 2}}{5} = \frac{12}{5}\sqrt{6}$$

$$= 5{\cdot}879 \text{ inches.}$$

Right-angled triangle.—When the included angle is a right angle, $B=90^\circ$ and $\sin 90^\circ=1$;

$$\therefore \text{area}=\tfrac{1}{2}ab.$$

Ex. 2. The sides of a right-angled triangle are 4·3 inches and 5·4 inches. Find the length of the perpendicular from the right angle on the hypotenuse.

$$\text{Hypotenuse}=\sqrt{4\cdot3^2+5\cdot4^2}=\sqrt{47\cdot65}.$$

$$\text{Area}=\tfrac{1}{2}\times4\cdot3\times5\cdot4=\tfrac{1}{2}p\sqrt{47\cdot65};$$

$$p=\frac{4\cdot3\times5\cdot4}{\sqrt{47\cdot65}}$$

$$=3\cdot36 \text{ inches}$$

Equilateral triangle.—In an equilateral triangle $a=b=c$ and each angle is 60°.

$$\therefore \text{Area}=\tfrac{1}{2}ac\sin B=\tfrac{1}{2}a^2\sin 60^\circ=\tfrac{1}{4}a^2\sqrt{3}$$

Ex. 3. Find the area of an equilateral triangle each side of which is 10 ft. long.

$$\text{Area}=\frac{10^2\sqrt{3}}{4}=\frac{173\cdot2}{4}$$

$$=43\cdot3 \text{ sq. ft.}$$

Area of a regular polygon.—If AB (Fig. 53) is one side of a regular polygon of n sides, the circle passing through the angular points is called the **circumscribed circle**. The circle touching all the sides of the figure is called the **inscribed circle**.

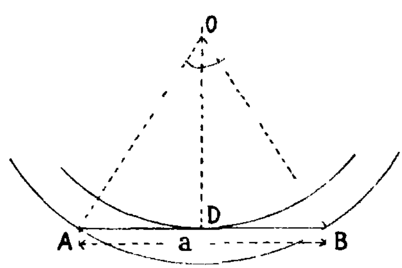

Fig. 53.—Area of a regular polygon.

The angle AOB is $\dfrac{360^\circ}{n}$, and if a perpendicular OD be drawn to side AB, then angle $AOD=\dfrac{180^\circ}{n}$. Denoting the length of the side AB by a,

$$\text{area of triangle } AOB=\tfrac{1}{2}AB\times OD$$

$$=\frac{a}{2}\times OD \quad\ldots\ldots\ldots\ldots (1)$$

If r denote the radius of the inscribed circle,

$$\text{area of triangle } AOB = \frac{a}{2} r$$

As
$$r = \frac{a}{2} \cot \frac{180^\circ}{n}, \quad \cdot \quad \text{(ii)}$$

$$\cdot \text{ area of triangle } AOB = \frac{a^2}{4} \cot \frac{180^\circ}{n}, \ldots \quad \text{(iii)}$$

and
$$\text{area of polygon} = \frac{na^2}{4} \cot \frac{180^\circ}{n} \quad \ldots \text{(iv)}$$

From (iv), the area of a polygon can be obtained when the length of one side is given

To obtain the area when the radius r is given, we may eliminate a from (iv) by means of (ii).

$$\text{Area of polygon} = nr^2 \tan \frac{180^\circ}{n}$$

To obtain the area of the polygon in terms of R, the radius of the circumscribed circle, we have from Fig 53,

$$OD = R \cos \frac{180^\circ}{n} \qquad \text{Also, } \frac{a}{2} = R \sin \frac{180^\circ}{n},$$

$$\text{area of polygon} = nR^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}$$

$$= \frac{nR^2}{2} \sin \frac{360^\circ}{n} \text{ (p. 32)}$$

Perimeter of polygon $= na = 2nr \tan \frac{180^\circ}{n} = 2nR \sin \frac{180^\circ}{n}$.

Ex 4 In a hexagon R is equal to the length of the side a,

$$\text{area} = \frac{6a^2}{2} \sin 60^\circ$$

$$= \frac{3\sqrt{3}a^2}{2}$$

Ex 5 Find the area of a regular pentagon in a circle of 4 inches radius

Here $n = 5$, $R = 4$;

$$\cdot \text{ area} = \frac{5 \times 16}{2} \sin 72^\circ = 40 \sin 72^\circ$$

$$= 40 \times 0{\cdot}9511 = 38{\cdot}044 \text{ sq. in}$$

Trapezium.—A four-sided figure such as $ABCD$ (Fig. 54), in which two sides AD and BC are parallel, is called a **trapezium.**

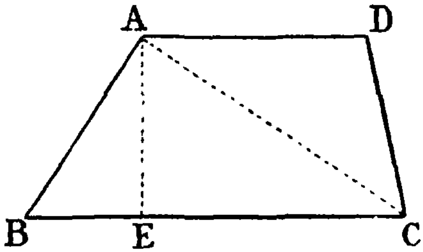

FIG 54 —Area of a trapezium

If a and b denote the lengths of AD and BC, and h the perpendicular distance AE between them, then, joining the points A and C by the line AC, the figure is divided into the two triangles ABC and ACD

$$\text{Area of triangle } ACD = \frac{1}{2}ah,$$

$$\text{,, \quad ,, \quad ,,} \quad ABC = \frac{1}{2}bh;$$

$$\text{area of } ABCD = \frac{1}{2}(a+b)h, \text{ or in words,}$$

area of a trapezium is one-half the sum of two parallel sides multiplied by the perpendicular distance between them.

EXERCISES. XXI.

1 The area of a rectangular field is 462 square yards, its length is 25 yards 2 feet; find its width.

2. Find the cost of enclosing a square field, area two acres, with a fence costing 3s. 6d per yard.

3. A public garden occupies two acres, and is in the form of a square. If a pathway goes completely round its inner edge, and occupies one-eighth of an acre, what is its width? [Acre = 4840 square yards.]

4. The area of a rectangular field is $\frac{2}{5}$ of an acre, and its length is double its breadth; determine the length of its sides.

5. In a quadrilateral the diagonal is 84 feet, and the two perpendiculars on it from the other two angles are 16 feet and 18 feet respectively; find the area.

6. Find the area of a triangle, base 625 links, height 1040 links [100 links = 22 yds.].

7. The length of each side of a hexagon is 12 feet; find its area.

8. The area of a hexagon is 286 437 square feet; find the length of a side.

9. Find the area of a triangle whose sides are 21, 20 and 29 inches respectively.

10. The three sides of a triangle are 15, 16 and 17 feet respectively; find its area

11. If the lengths of the sides of a triangle be 242, 1212 and 1450 yards, show that the area is 6 acres.

12 Find the area of a triangular field ABC from the following measurements on the Ordnance Survey of 25 inches to the mile: AC 4·1 inches, perpendicular from B on AC 1·59 inches Calculate the area of the triangle ADC from the three sides, AB measuring 3 3 inches and BC 2 inches. Express the mean of the two in acres.

13. The diagonal of a rectangular field is $6\frac{1}{2}$ chains What is the length and width if the area is $1\frac{1}{2}$ acres? [1 chain = 22 yds]

14 Find the area of a quadrilateral of which the diagonal is 1274 feet and the perpendiculars upon it from the opposite angles 550 and 583 feet respectively

15. The perimeter of a square field is 588 yards and of another 672 yards. Find the perimeter of a third equal in area to the other two together.

16. Find the area of a quadrilateral $ABCD$ in which the sides AB, BC, CD, DA, and diagonal AC are 25, 60, 52, 39 and 65 respectively.

17 Each side of a rhombus is 120 yards and two of its opposite angles are each 60°; find the area

18. A field in the form of a trapezium has its parallel sides 10 chains 30 links and 7 chains 70 links If the area be 6 acres 3 roods, find the length of the shortest way across the field

19. The parallel sides of a trapezium are 5 chains 15 links and 3 chains 85 links respectively, the perpendicular distance between them is 15 chains; find the area.

20. The side of an equilateral triangle is 20 feet; find the numerical value of the radius of the circle circumscribing the triangle.

21. Regular polygons of 15 sides are inscribed in and circumscribed about a circle whose radius is one foot; show that the difference of their areas is nearly 20 square inches.

Circle.—The following rules are important:

$$\textbf{Circumference} = 2\pi r, \text{ or } \pi d$$

$$\textbf{Area} = \pi r^2, \text{ or } \frac{\pi}{4}d^2,$$

where r denotes the radius and d the diameter of the given circle

Annulus or circular ring.—If the external radius is R and internal radius r

$$\textbf{Area of annulus} = \pi(R^2 - r^2) = \pi(R + r)(R - r),$$

or
$$\frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(D + d)(D - d),$$

where D and d denote the external and internal diameters respectively

Ellipse -If $2a$ and $2b$ denote the lengths of the major and minor axes respectively, then

$$\textbf{circumference} = \pi(a + b), \textbf{ approx.}; \quad \textbf{area} = \pi ab.$$

Ex 1 Find the radius of a circle equal in area to that of an ellipse whose axes are 21 ft and 14 ft

Let r denote the radius of the circle

Then, area of circle $= \pi r^2 = \pi\left(\frac{21}{2} \times \frac{14}{2}\right)$;

$$\therefore r = \sqrt{\left(\frac{21}{2} \times 7\right)} = \sqrt{\frac{147}{2}}.$$

$$\log r = \tfrac{1}{2}(\log 147 - \log 2) = 0{\cdot}9331 = \log 8{\cdot}572;$$
$$r = 8{\cdot}572 \text{ ft}$$

Area of sector of a circle.—The **area of the sector of a circle AOE is one-half the product of the angle in radians and the square of the radius.**

Let A denote the area,

$$A = \frac{\theta r^2}{2}.$$

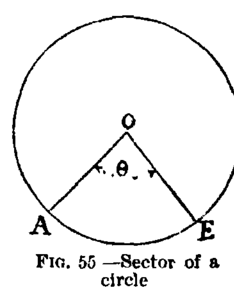

FIG. 55 —Sector of a circle

If N denotes the number of degrees in the angle AOE, then, as the sector is simply a fractional part of the circle,

$$\text{Length of arc } AE = \frac{N}{360^\circ} \times 2\pi r$$

$$\text{Area of sector} = \frac{N}{360^\circ} \times \pi r^2.$$

The two following theorems are important and may be verified by drawing the figures to scale:

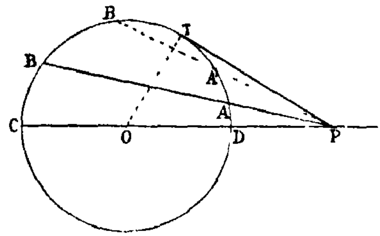

FIG 56.—$PT^2 = PA \times PB$

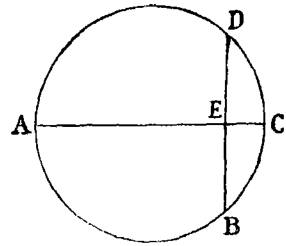

FIG 57—$AE \cdot EC = DE^2$

(i) From any point P outside a circle draw two lines—one which touches, or is a tangent to, the circle; the other cutting it in two points A and B Then $PT^2 = PA \times PB$

(ii) If two straight lines within a circle, such as AC and BD, cut one another at a point E, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other, i e $AE \cdot EC = DE \cdot EB$

If one line such as AC passes through the centre of the circle and the other is perpendicular to AC, then $DE = EB$, $\therefore$

$$AE \cdot EC = DE^2.$$

Segment of a circle.—Any chord of a circle, which is not a diameter, such as AB (Fig 58), divides the circle into two parts, one greater and one less than a semicircle

If C is the centre of the circle of which the given arc ADB forms a part, then **the area of the segment ADB is equal to the difference between the area of the sector $CADB$ and the triangle ABC**

Length of arc ADB (Huygens' Approximation).—The length of the arc ADB may be found approximately by the rule.—**Subtract the chord of the arc from eight times the chord of half the arc and divide the result by 3.**

$$\text{Length of arc } ADB = \frac{8AD - AB}{3} = \frac{8a - c}{3},$$

where a denotes the length of the chord AD (of half the arc) and c the length of AB (chord of the whole arc)

It will be found that results may be obtained by this rule to a fair degree of accuracy, but the angle must not be too large, *i.e.* the rule should not be used for angles greater than 90°. Thus, for 80°, the rule gives 1 3953, the accurate value is 1·3953. For an angle of 167° the length obtained by the rule is in error by 1%.

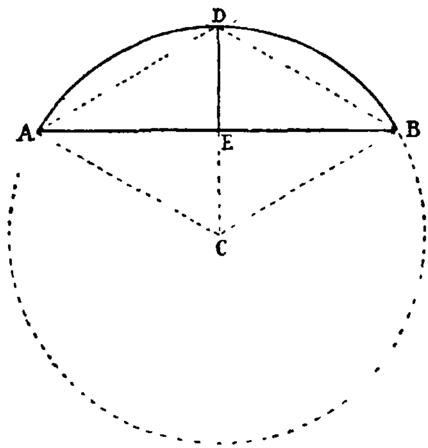

FIG. 58 — Segment of a circle

Area of segment.—If h denote the height ED (Fig. 58), the area of the segment is approximately

$$\frac{h^3}{2c}+\frac{2}{3}ch, \text{ or } \frac{h}{6c}(3h^2+4^2c).$$

Chord of a circle.—The chord of an arc, c, and the chord of half the arc, a, may be expressed in terms of the height, h, thus, produce DE to cut the circumference of the circle in a point F.

Since $$AE\times EB=FE\times ED;$$

$$\therefore \left(\frac{c}{2}\right)^2=h(2r-h);$$

$$\therefore c^2=4h(2r-h) \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Also, $$a^2 = \frac{c^2}{4} + h^2;$$

$$\therefore c^2 = 4(a^2 - h^2).$$

Substitute this value in (i);

$$\therefore a^2 = 2hr. \qquad \qquad \text{......(ii)}$$

Ex. 2. Three vertical posts are placed at intervals of one mile along a straight canal, each rising to the same height above the surface of the water. The straight line joining the tops of the two extreme posts cuts the middle post at a point 8 inches below the top; find, to the nearest mile, the radius of the earth.

As the two distances and the radius are large compared with 8 inches, the chord may be taken to be of the same length as the arc;

$$a = \frac{c}{2} = 5280 \times 12 \text{ inches.}$$

Hence, if r denote the radius,

$$2rh = a^2,$$

$$\text{or } r = \frac{(5280 \times 12)^2}{2 \times 8} \text{ inches}$$

$$= \frac{(5280 \times 12)^2}{16 \times 5280 \times 12} = 3960 \text{ miles}$$

Area of a segment of a parabola.—The area of a portion of a parabola such as ABC (Fig. 59) is two-thirds the product of the base and the height,

$$\text{area of parabola} = \tfrac{2}{3}ab.$$

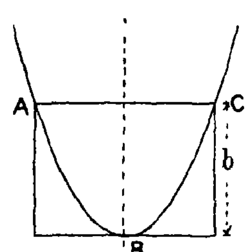

FIG. 59.—Area of segment of a parabola.

Ex. 3. Find the area of the segment of a circle, chord 40 in., height 6 in. What would be the area of a parabolic segment having the same dimensions?

$$\text{Area} = \frac{h^3}{2c} + \frac{2}{3}ch$$

$$= \frac{6^3}{80} + \frac{2}{3} \times 40 \times 6 = 2 \cdot 7 + 160$$

$$= 162 \cdot 7 \text{ sq. in.}$$

The area of a parabolic segment is $\frac{2}{3}$ (product of chord and height)

$$= \tfrac{2}{3} \times 40 \times 6 = 160 \text{ sq. in.}$$

Area of an irregular figure.—When the boundaries of an irregular figure consist of straight lines, the area may be obtained by dividing the figure into a number of triangles, rectangles, etc. The sum of the areas of all the simple figures, into which the given figure has been divided, will be the area required. When one or more of the boundaries of a given figure consist of curved lines, the area may be found by one of the following methods explained in the elementary course the student is already supposed to have taken (*a*) by using a planimeter, (*b*) using squared paper, (*c*) by weighing, (*d*) by mid-ordinate rule

In addition to the above methods there are, amongst others, the trapezoidal rule, and the two important rules of Simpson and Weddle (p. 405).

Planimeter.—The planimeter is an instrument adapted for estimating rapidly and accurately the area of any figure. There are many forms in general use to which various names—Hatchet, Amsler, etc—are given Of these the more accurate forms are mostly modifications of the Amsler planimeter

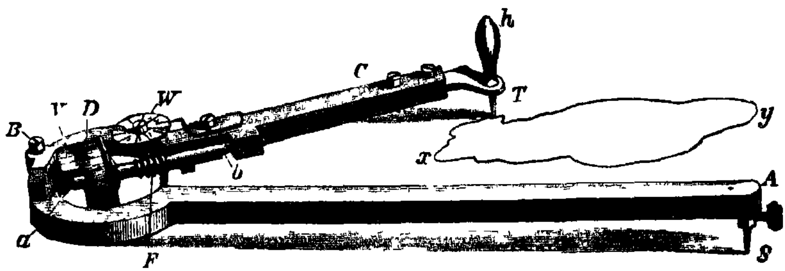

FIG. 60 —Amsler planimeter.

Amsler planimeter.—One form of the instrument is shown in Fig. 60, and consists of two arms *AB* and *BC*, pivoted together at a point *B* The arm *BA* is fixed at some convenient point *s* The other arm *BC* carries a tracing point *T*. This tracing point is passed round the outline of the figure, the area of which is required The arm *BC* carries a wheel *D*, the rim of which is usually divided into 100 equal parts, about which it turns as an axis and records by

its revolution the area of the figure traced out by T From its construction it is obvious that the revolving wheel registers only the motion which is perpendicular to the moving arm on which it revolves.

When the instrument is in use, the rim of the wheel rests on the paper, and, as the point T is carried round the out line of the figure, the wheel, by means of a spindle rotating on pivots at a and b, gives motion to a small worm F, which in turn rotates the dial W

One rotation of the wheel corresponds to one-tenth of a revolution of the dial. A vernier, V, is fixed to the frame of the instrument, and a distance equal to 9 scale divisions on the rim of the wheel is divided into ten on this vernier The readings on the dial are indicated by means of a small finger, or pointer, shown in Fig. 60. If the figures on the dial indicate units those on the wheel will be $\frac{1}{10}$ths, as each of these is subdivided into 10, the subdivisions indicate $\frac{1}{100}$ths. Finally, the vernier, V, in which $\frac{9}{100}$ of the wheel is divided into 10 parts, enables a reading to be made to three places of decimals.

To obtain the area of a figure, the fixed point s is set at some convenient point which may be outside or inside the area to be measured and the point T at some point in the periphery of the figure Note the reading of the dial and wheel Carefully follow the outline of the figure until the tracing point T again reaches the starting-point a second time, and again take the reading. If the fixed point s has been chosen *outside* the given area, all that is now necessary is to multiply the difference between the two readings by a certain constant to obtain the area of the figure; the value of the constant may be found by using the instrument to obtain a known area, such as a square, or circle of known radius If the fixed point s had been chosen inside the figure it is possible to clamp the joint B of the instrument so that whilst T describes a circle, the indicating wheel shall always move on the paper perpendicular to the plane of its rim, and consequently register no rotation in any part of its course The circle which T thus describes is called the zero circle, and

its area (marked on the instrument) must be added to the indication of the instrument in order to obtain the measure of a given area.

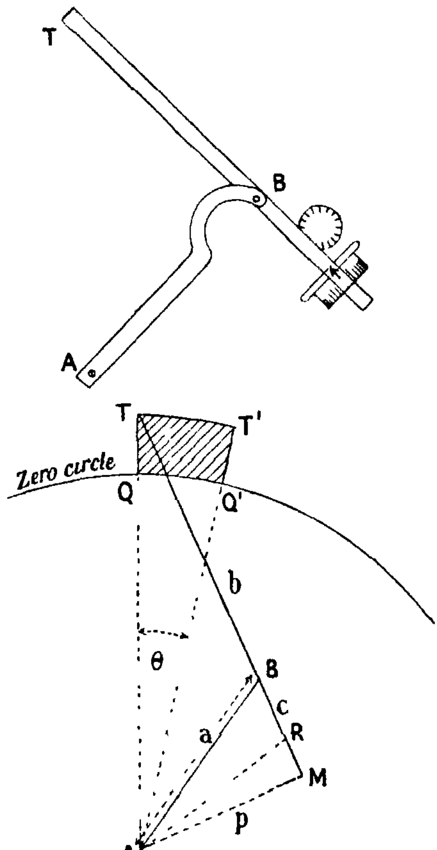

FIG 61

T is the tracing point (Fig. 61) and A the fixed point. **When** AR is perpendicular to TM and the joint at B is locked (*i.e.* does not turn), the point T describes a circle, called the **zero circle**, about A as centre. The indicating wheel under these conditions remains stationary.

Let $\qquad AT=r$ and $AQ=r_0$

The shaded area $\qquad QTT'Q'=\frac{\theta}{2}(r^2-r_0^2).$

Draw AM perpendicular to and meeting TB produced in M.

Let $\quad AB=a, \quad BT=b, \quad BR=c, \quad RM=m$

Then, from the right-angled triangle AMT, AT^2 or

$$r^2=AM^2+MT^2.$$

But $\quad AM^2=a^2-(c+m)^2=a^2-(c^2+2cm+m^2),$

and $\quad MT^2=(b+c+m)^2=b^2+c^2+m^2+2bc+2bm+2cm\,;$

$$\therefore r^2=a^2+b^2+2b\,(c+m)$$

Similarly, when AR is perpendicular to TR, from the right-angled triangle ART, we obtain

AQ^2 or $r^2_0=AR^2+RT^2$

$$AR^2=a^2-c^2.$$

Also $\quad RT^2=(b+c)^2=b^2+2bc+c^2\,;$

$$\therefore r_0^2=a^2+b^2+2bc\,,$$

$$\therefore \frac{\theta}{2}(r^2-r_0^2)=\frac{\theta}{2}\{a^2+b^2+2b(c+m)-(a^2+b^2+2bc)\}$$

$$=\theta bm$$

Now the linear speed of the tracing point $T=\omega AT=\omega r$.

Speed of *sliding* of wheel $=\omega AM$

Speed of *turning* of wheel $=\omega m$

As the tracing point T moves along TT', the wheel registers $\theta\times m$.

And, as the tracing point moves along QQ', the wheel remains stationary.

Also, the motions given to the wheel as the tracing point moves over QT and $T'Q$, are equal in amount but opposite in direction

Hence, in tracing the boundary of the shaded area, the wheel records a motion of $\theta\times m=\frac{\text{area}}{b}$,

$$\text{area}=b\theta m=b\times\text{motion of wheel}$$

The tracing point T is usually carried by a bar which can slide in a sleeve carrying the point B, and the adjustment is made by altering the position of B.

Simpson's Rule.—When an odd number of ordinates are given, except in the special case of 7 ordinates, probably the most acccurate rule that can be used is **Simpson's First Rule.** As this rule is so important it is usually referred to simply as Simpson's Rule. Except where otherwise expressed the following exercises are supposed to be solved, as in the following example, by using Simpson's Rule:

Ex. 4. An irregular figure has the following ordinates (in feet):

3·5, 4 75, 5 25, 7·5, 8·25. 14·75, 6, 9 5, 4

The common interval being 2·5 ft., find the area

$$\text{Area} = \frac{S}{3}(A + 4B + 2C),$$

where S denotes the common interval, A the sum of extreme ordinates, B the sum of the even ordinates, C the sum of the odd ordinates;

sum of extreme ordinates $= 3\ 5 + 4 = 7\ 5$;

$\therefore$ sum of even ordinates $= 4\ 75 + 7\ 5 + 14\ 75 + 9\ 5 = 36{\cdot}5$;

$\therefore$ sum of odd ordinates $= 5{\cdot}25 + 8\ 25 + 6 = 19\ 5$

$$\text{Area of figure} = \frac{2\ 5}{3}(7\ 5 + 4 \times 36\ 5 + 2 \times 19\ 5) = 160\ 41.$$

Mean ordinate.—The product of the mean ordinate and the length of the line assumed as the base of an irregular figure gives its area. Hence, in order to obtain the mean ordinate in any of the preceding cases, it is only necessary to divide the calculated area by the length Thus, in the preceding example, as the line EF is 20 feet;

$$\text{mean ordinate} = \frac{160{\cdot}41}{20}$$

$$= 8\ 02 \text{ feet.}$$

EXERCISES. XXII.

1 Find the perimeter and the radius of a circle the area of which is 5·3093 square feet.

2. The area of a semicircle is 13013 square feet; find its total perimeter.

3. One circle is described *about* and a second is inscribed *within* a regular hexagon length of side 1 foot; find the area between the two circles.

4. The side of a regular hexagon is 2 feet; find the radius of a circle equal to it in area.

5. The radius of a circle is 33.5 feet; find the area of a sector enclosed by two radii and an arc 133.74 feet in length.

6. Find the length of an arc which subtends an angle of 60° in a circle whose radius is 100 feet.

7. The length of an arc subtending an angle of 60° is 11 feet; find the radius of the circle.

8. The area of a trapezoidal field is $4\frac{1}{2}$ acres, the perpendicular distance between the parallel sides is 120 yards, and one of the sides is 10 chains; find the other.

9. The minute hand of a clock is 10 inches long; find the area which it describes on the clock face between 9 a.m. and 9.35 a.m.

10. The radius of circle is 8 feet; find the area of a sector of the circle, the angle of which is 36°.

11. Find the radius of a circle such that the area of a sector corresponding to an angle of 90° may be 181.16 square feet.

12. Find the radius of a circle in which an arc 15 inches long subtends at the centre an angle containing 71° 36′.

13. The side of an equilateral triangle is 20 feet; find the radius of the circle circumscribing the triangle.

14. The interior diameter of a circular building is 51 feet and the thickness of wall 2 feet. What is the area occupied by the wall?

15. A road 10 feet wide has to be made round a circular plot of ground 75 yards diameter; find the cost of the road at 4s. per square yard.

16. The diameters of the piston and air-pump of an engine are as 2 : 1·2; find the diameter of the air-pump when the area of the piston is 1134.1 square inches.

17. Find the length of an arc of a circle of radius 20 feet subtending a certain angle at the centre, when the length of an arc of a circle of radius 4 feet, subtending three times the former angle at the centre, is 9 feet.

18. If three equal circles whose common radius is 12 inches touch each other, what is the area enclosed between them?

19. A circular grass plot is surrounded by a ring of gravel b feet wide; if the radius of the circle, including the ring, be a

feet, find the relation between a and b, so that the areas of grass and gravel may be equal.

20 Find the expense of paving a circular court 80 feet in diameter, at 3s. 4d. per square foot, leaving in the centre a space for a fountain in the shape of a hexagon, each side of which is a yard.

21. The area of an equilateral triangle is 17320·5 square feet. About each angular point as centre, a circle is described with radius equal to half the length of a side of the triangle. Find the area of the space included between the three circles.

22. The semi-ordinates of the load water plane of a vessel in feet are respectively 0 1, 5, 11·6, 15 4, 16 8, 17, 16·9, 16·4, 14·5, 9·4 and 0 1; the common interval is 11 feet. Find the area of the plane in square feet.

23 The half-ordinates of a water plane are 15 feet apart, and their lengths are respectively · 1·9, 6·6, 11, 14·5, 17·4, 19·4, 20·5, 20 8, 20 3, 18·8, 15 8, 10 6 and 2 6 feet. Find the area of the plane.

24 The semi-ordinates of the load water plane of a vessel are 0 2, 3·6, 7 4, 10, 11, 10 7. 9 3, 6·5 and 2 feet respectively, and they are 15 feet apart What is the area?

25 The half-ordinates of the load water plane of a vessel are spaced 18 feet apart, and their lengths are 0 6, 3·4, 7 1, 11 4, 16 0, 20 3, 24 0, 26 8, 28 8, 30 0, 30 5, 30 5, 30 0, 28 9, 27·0, 24 3, 21·1, 17 2, 12 7, 7 7 and 3 0 feet respectively. Calculate the total area of the plane in square feet.

26 The ordinates of a curved figure in inches are, 2·6, 3 5, 3·66, 3 63, 3 37, 2 85, 2 4, 2 1, 1 89, 1 74, 1·6, 1·38, 0 49; common interval $\frac{1}{2}$ inch. Find the area

27. The length of an indicator diagram is 4 inches, the end ordinates are 1, 0 22, and the other ordinates are 1, 0·82, 0 71, 0·55, 0 45, 0·38, 0 33, 0 29 and 0 26 inches respectively. The scale of pressure is 60 lbs per square inch to one inch. Find the mean pressure (i) by the common rule, (ii) by Simpson's rule.

28. The half-ordinates of the midship section of a vessel are 22 3, 22·2, 21 7, 20·6. 17·2, 13·2 and 8 feet in length respectively. The common interval between consecutive ordinates is 3′ between the 1st and 5th ordinates and 1′ 6″ between the 5th and 7th ordinates. Calculate the total area

29 The half-ordinates of the midship section of a vessel are 12·8, 12 9, 13, 13, 13, 12 9, 12 6, 12, 10 5, 6 and 1·5 feet respectively; the common distance between the ordinates is 18 inches. Find the area.

CHAPTER X.

MENSURATION OF SOLIDS

Prism.—The base of the right prism (Fig 62) is the rectangle $ABCD$ If l denote the length AB, and b the width BC, then the area of the base is $b \times l$ If the thickness, or height, BE, be denoted by h, then, if V denote the volume of the prism,

$$V = bl \times h,$$

or **volume**

$$= \textbf{(area of base)} \times \textbf{(height)}$$

The surface of the solid consists of six rectangles If S denote the total surface, then

$$S = 2(bl + hl + bh)$$

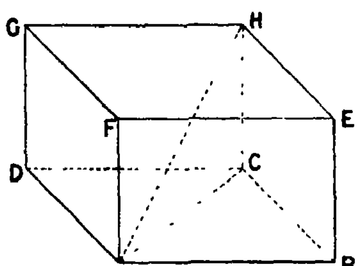

FIG 62.—Rectangular prism

If A be joined to C, the triangle ACB is a right angled triangle and

$$AC^2 = AB^2 + BC^2 ,$$

$$AC = \sqrt{b^2 + l^2}.$$

The line AH joining two opposite corners A and H is called a *diagonal of the solid.*

And $$AH^2 = AC^2 + CH^2 = b^2 + l^2 + h^2 ,$$

$$\therefore AH = \sqrt{b^2 + h^2 + l^2}.$$

Ex. 1. **The length, width, and height, of a rectangular prism are 5, 3, and 2 feet respectively. Find the volume, the surface, and the length of a diagonal, of the solid.**

$$V=5\times3\times2=30 \text{ cubic feet}$$
$$S=2(5\times3+2\times5+2\times3)$$
$$=62 \text{ square feet}$$
$$\text{Length of diagonal}=\sqrt{5^2+3^2+2^2}$$
$$=6\cdot164 \text{ feet.}$$

Oblique prism.—The volumes of all prisms, so long as they have the same, or equal, bases and the same altitude, are equal Thus, in Fig. 63, an oblique prism $ADCGFBE$ is shown. By drawing CN and DH perpendicular to DC, and NP and HM parallel to BF, wedge-shaped pieces are obtained. Assuming the wedge-shaped piece $GCNPFB$ transferred to the left as indicated, the oblique prism becomes a right prism, and thus, as before, the volume of the prism is equal to the area of the base multiplied by the altitude

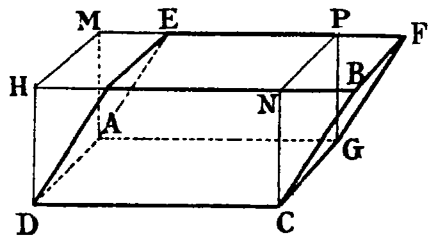

FIG 63 —Oblique prism.

Cube.—When the three dimensions of length, breadth, and height, are all equal and all the angles right-angles, the solid is called a **cube**, or a cube may be defined as contained by six plane faces all of which are squares. If a denote the length of the edge of the solid, then

$$V=a^3,\quad S=6a^2.$$
$$\text{Diagonal of solid}=\sqrt{3a^2}=a\sqrt{3}$$

Cylinder.—The base of a prism may consist of any plane closed curve, and it has been seen that the **volume is the product of area of base and height.** When the base is a circle of radius r, and the height (or length) of the cylinder is denoted by h, the volume V and curved surface S are obtained by using the rules,

$$V=\pi r^2h, \quad\text{.......................}(i)$$
$$S=2\pi rh \quad\text{.......................}(ii)$$

Total surface.—To obtain the total surface, the areas of the two ends must be added to (ii) This gives

$$\text{Total surface}=2\pi rh+2\pi r^2=2\pi r(h+r).$$

Weight.—The weight of the solid is the volume multiplied by weight of unit volume. This may be written $W = Vw$, where w is the weight of unit volume.

Hollow circular cylinder.—If V is the volume, S the curved surface of a hollow cylinder, external radius R, internal radius r, and height h, then

$$V = \pi(R^2 - r^2)h, \qquad \text{(i)}$$

$$S = 2\pi Rh + 2\pi rh$$

$$= 2\pi(R + r)h. \qquad \text{(ii)}$$

The thickness of the material of a cylinder is $R - r$, and dividing (i) by (ii)

$$\frac{V}{S} = \tfrac{1}{2}(R - r).$$

Oblique cylinder.—In the preceding paragraphs what are called right cylinders have been assumed, viz., the sides of the prism are at right angles to the plane of the base, but the preceding rules apply equally to oblique prisms, when S and A are as follows

S = area of curved surface together with the sum of the areas of the two ends.

V = (area of base) × (altitude)

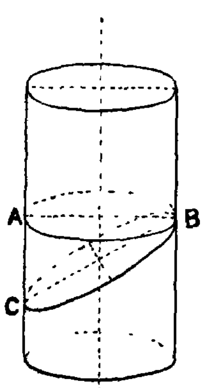

FIG 64.—Cylinder

Cross section.—The term *cross section* is generally used to denote the section of a right cylinder, or a right prism, by a plane perpendicular to its axis Thus, the term *radius of a cylinder* is simply a shortened expression for the radius of a perpendicular cross section. If AB (Fig 64) indicates the cross section of a circular cylinder (which is a circle), any oblique section such as BC will be an ellipse Also the area of an oblique section BC multiplied by the cosine of the included angle will give the area of the cross section, i.e

$$AB = BC \cos ABC.$$

Ex 2. The diameter of a right circular cylinder is 3 inches. There is a section making an angle of 20° with the cross section. What is its area?

$$\text{Area of cross section} = \pi\left(\tfrac{3}{2}\right)^2.$$

As $$AB = BC \cos ABC,$$

$$\therefore \text{ area of } BC = \frac{\text{area of } AB}{\cos 20^\circ} = \frac{\pi \times \left(\frac{3}{2}\right)^2}{0\ 9397}$$

$$= \frac{9\pi}{4 \times 0\ 9397} = 7{\cdot}523 \text{ sq in}$$

Ex. 3 A prism has a cross section of 50 32 square inches. There is a section making an angle of 70° with the cross section. What is its area?

$$\text{Area} = \frac{50{\cdot}32}{\cos 70^\circ} = \frac{50\ 32}{3420}$$

$$= 147\ 2$$

EXERCISES. XXIII

1. In a circular cylinder, volume V, curved surface S, height h, and radius of base r, weight of unit volume w

(i) If $r=8$ ft, $h=8$ ft, find S and V

(ii) If $S=66\ 759$ sq ft, and $V=70{\cdot}93$ cub ft, find r

(iii) Find W if $r=6$ in, $h=20$ in, $w=0\ 3$ lbs. per cub. in.

(iv) $V=5497{\cdot}8$ cub ft, $r=2\frac{1}{2}$ ft; find h

2 The length, width and thickness of a rectangular block are 9 6, 13 2 and 14·3 inches respectively Find the volume, the surface, and the length of a diagonal of the solid

3 If V is the volume, S the curved surface of a hollow cylinder, external radius R, internal radius r, height or length h and w is the weight of unit volume—

(i) If $R=5$ in, $r=3$ in., $h=8$ in, find S and V; also find W if $w=0{\cdot}26$ lbs per cub in

(ii) If $V=36{\cdot}67$ cub. ft, $S=220$ sq ft, find $R-r$.

(iii) If $W=8\ 2$ tons, $R=9$ in., $r=5$ in $w=0{\cdot}29$ lbs. per cub. in., find h.

4 Find the total surface, also the volume, of a hexagonal prism, height = 8 ft., base a regular hexagon, with a side of length = 3 ft.

5. The volume of a square bar of copper 40 feet in length is 1 cubic foot. If the greatest exact cube is cut from the bar, what will be its weight? (1 cub. in copper = 0·3192 lbs)

6. Find the weight of a wrought iron cylinder, outer circumference 10 ft. 7·3 in., height 3 ft. 6 in., thickness of metal $\frac{1}{2}$ inch. (1 cub. in. weighs 0 28 lbs.)

7. What weight of water will fill a hose pipe 2 in. bore and 90 ft. long? (1 cubic foot of water weighs 62·3 lbs.)

8. Find the volume and weight of 6 ft length of a cast-iron pipe, outer diameter 12·5 in. and thickness of metal $\frac{7}{8}$ in. (1 cubic in. weighs 0·26 lbs.)

9. Find the surface and volume of hollow cylinder, height 12 in., internal and external radii of base 4 in and 6 in. respectively.

10. The base of a prism is a triangle, sides 17, 25 and 28 ft. respectively. The volume of the prism is 4200 cub. ft. What is its height?

11. Find the internal width of a square bottle to hold a quart of water when the depth is 6 inches (1 gallon of water weighs 10 lbs.)

12. A section of a stream is 10 ft wide and 10 inches deep; the mean flow of the water through the section is 3 miles an hour, find how many gallons of water flow through the section in 24 hours.

13. Determine the number of cubic yards in a bank of earth on a horizontal rectangular base 60 ft. long and 20 ft broad, the four sides of the bank sloping up to a ridge at an angle of 40° to the horizon.

14. The water in a rectangular reservoir is $9\frac{1}{2}$ ft. deep and covers an area of 5390 square yards. In what time can the water be emptied by a pipe 5 inches in diameter, through which the water runs at the rate of 17 miles per hour?

15. A cylindrical vessel 16 feet diameter, 20 feet long, is filled with water at 210° C., what is the weight of water in tons? [Water is 17·2 per cent greater in volume at 210° C than when cold.]

16. A prism has a cross-section of 50·32 square inches. There is a section making an angle of 20° with the cross section; what is its area?

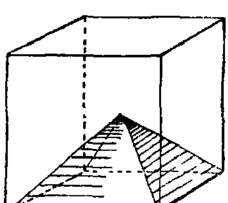

FIG. 65.—Volume of a pyramid.

Pyramid.—The volume of a pyramid is

$$\tfrac{1}{3}(\text{area of base}) \times \text{height}$$

This important result may be easily derived from the known volume of a cube. By joining each angular point of a cube to the centre (Fig. 65) six equal pyramids are formed. The base of each pyramid is one of the faces of the cube. Hence, the volume of each pyramid is one-sixth of the cube.

If a denote the length of each side of the cube, then h the height of the pyramid is $\frac{a}{2}$.

$\therefore$ Volume of pyramid $= \frac{1}{6}a^3$

$$= \tfrac{1}{3}a^2 \times \frac{a}{2}.$$

Hence, **volume of pyramid $= \frac{1}{3}$(area of base) $\times$ height,**
or volume of a pyramid is one-third that of a prism on the same base and the same altitude.

If A denote area of base, then volume is given by

$$V = \tfrac{1}{3}Ah,$$

a result which applies both to right and oblique pyramids.

Surface of a pyramid.—The surface, or area, of a pyramid consists of the **lateral surface**, this is the area of a number of triangles which form the faces, or sides, of the figure, together with the **area of the base** (which may be any polygon). In a right pyramid, if the polygon forming the base be regular, each of the faces ABO, BCO, etc., of the solid (Fig. 66) consist of equal isosceles triangles. If a denote the length of the edge AB, h the height OP, and l the slant height OQ, the slant height is the same for each triangle only when a circle can be described touching *each* side of the polygonal base. If the radius of such a circle be r and if h be the height of the pyramid, then

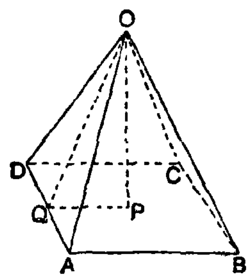

FIG. 66.—Surface of a pyramid.

$$\text{slant height, } l = \sqrt{r^2 + h^2},$$

$$\text{area of each triangle} = \tfrac{1}{2}(a \times l).$$

The slant surface of a right pyramid whose base is a regular polygon of n sides each equal to a is $\frac{1}{2}nal$.

$\therefore$ **the lateral surface of a pyramid equals half the perimeter of the base multiplied by the slant height.**

When a and h are given, $\quad l = \sqrt{h^2 + \left(\frac{a}{2}\right)^2}.$

Cone.—A cone is the solid bounded by an area, and by lines passing through the successive points of the boundary of that area and a fixed point outside the plane of the given area. The area usually consists of a circle, or an ellipse, and

the preceding rules for volume and surface of a pyramid are used. When the area is circular and the given point is perpendicularly above the centre and at a distance h from it, if l (Fig. 67) denote the length AC, then

$$\text{the curved surface} = \tfrac{1}{2}(2\pi r)l = \pi r l.$$

When the altitude h and the radius of the base are given, from the right-angled triangle CBA,

$$l = \sqrt{h^2 + r^2}.$$

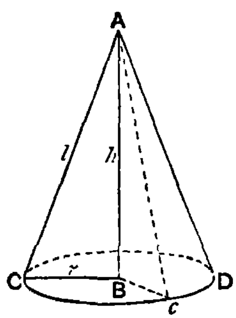

FIG 67 —Curved surface of cone

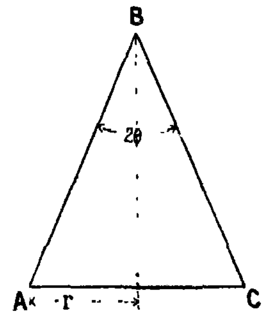

FIG 68 —Vertical angle of cone

If S denotes the curved surface and V the volume of the cone, then

$$S = \pi r l,$$

$$\text{total surface} = \pi r l + \pi r^2 = \pi r(l + r),$$

$$V = \tfrac{1}{3}\pi r^2 h$$

Generally, the volume of a cone, whether right or oblique, is $\frac{1}{3}$ (area of base × height)

Vertical angle.—If the vertical angle of a cone (Fig 68) be denoted by 2θ, then

$$r = h \tan\theta, \quad l = h \sec\theta.$$

Ex 1. Find the curved and the whole surface, the volume and vertical angle of a cone, when $r = 45$ in , $h = 48$ in

Here $\quad l = \sqrt{48^2 + 45^2} = \sqrt{4329}$

$= 65\ 8$ in.;

$S = \pi \times 45 \times 65\ 8 = 9302$ sq in ,

total surface $= \pi \times 45(65\ 8 + 45)$ sq. in.

$= 108{\cdot}8$ sq. ft.,

$$V = \frac{\pi}{3} \times 45^2 \times 48 \div 1728 = 58{\cdot}91 \text{ cub. ft.}$$

Vertical angle.—We have

$$\tan\theta = \frac{r}{h} = \frac{45}{65\cdot 8} = 0\cdot 6839\,;$$

$$\therefore\ \theta = 34^\circ\ 22',$$

$$\therefore\ \text{vertical angle} = 68^\circ\ 44'.$$

The curved surface of a right circular cone may also be obtained as follows:—Let a piece of thin paper be made to cover the surface of a cone exactly; then, when opened out, it will form a sector of a circle of radius equal to l. The length (Fig. 69) of the arc $CD = 2\pi r$, the area of sector is one half the product of the arc and the radius;

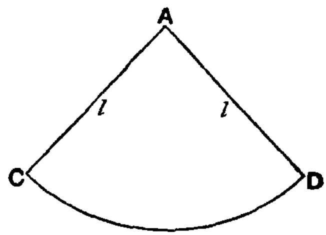

FIG. 69

$$\therefore\ \text{area of sector} = \tfrac{1}{2}\times 2\pi r\times l = \pi r l.$$

Frustum of a right pyramid on a regular base.—Each of the faces such as $ABCD$ of the frustum of a pyramid (Fig. 70) is a trapezium, and the area of each trapezium will be half the sum of the parallel sides, AB and CD, multiplied by the slant distance between them, and by the number of faces.

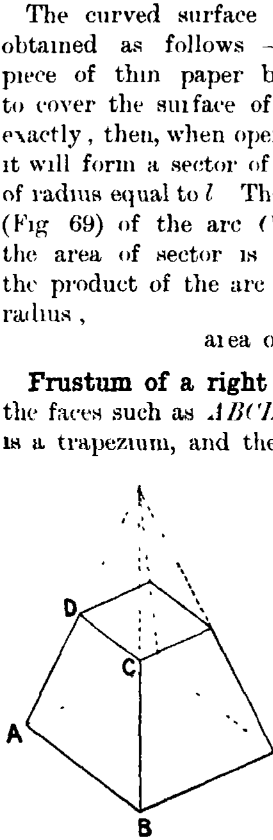

FIG. 70.—Frustum of a pyramid.

In the frustum of a pyramid on a square base (Fig. 70) let a denote the length of each side of the base, b the length of each side of the other end, l the slant height of the frustum.

Each face $ABCD$ is a trapezium, the lengths of the parallel sides a and b.

$$\therefore\ \text{Area } ABCD = \tfrac{1}{2}(a+b)l.$$

As there are four such trapeziums in the lateral surface S, we have $S = 2(a+b)l$, or

slant surface = ½(sum of perimeters of ends) × (slant height). ...(1)

The **total surface** would obviously be the **lateral surface together with the areas of the two ends.**

If h denote the altitude of the frustum, then the volume is given by

$$V=\tfrac{1}{3}h(a^2+b^2+ab);$$

or we may denote by A_1 the area of the base and by A_2 the area of the face parallel to it, then

$$V=\tfrac{1}{3}h(A_1+A_2+\sqrt{A_1A_2}) \quad \ldots \quad \text{(ii)}$$

The base of a pyramid may be any polygon, and the rule (i) may be used for any right regular frustum, *i e* **to the sum of the areas of the two ends add the square root of their product and multiply the result by one-third the altitude.**

Frustum of a cone.—A circular cone is merely a special case in which the base of a pyramid is a circle, and the preceding rules given by (i) and (ii) apply.

$$S=\pi(R+r)l, \quad \ldots \quad \text{(iii)}$$

$$V=\frac{h}{3}(\pi R^2+\pi r^2+\sqrt{\pi^2R^2r^2})$$

$$=\frac{\pi h}{3}(R^2+r^2+Rr) \quad . \quad \text{(iv)}$$

When the cutting plane passes through the vertex of the cone, r is zero, and putting $r=0$ in (iii) and (iv), the formulae for the surface and volume of a cone are obtained

The expressions dealing with the surface and volume of a frustum are of great use in calculations But it is quite unnecessary to attempt to commit them to memory A frustum may be considered as part of a whole, and by the subtraction of the surface and volume of the part removed the results for the frustum may be obtained Both methods of calculation are shown in the following example.

Ex. 2 Find the curved surface and volume of the frustum of a cone whose top and bottom diameters are 4 and 6 inches and the slant height 8 inches What is the surface and volume of the cone of which this frustum forms a part?

Here $R=3$, $r=2$, $l=8$;

$$\therefore S=\pi(3+2)8=40\pi$$

$$=125{\cdot}71 \text{ sq in.}$$

First obtain the height, h, of the frustum ;

$$\therefore h=\sqrt{8^2-(3-2)^2}=\sqrt{8^2-1^2}=\sqrt{9\times 7}=7\ 936 \text{ in.},$$

Then $V=\dfrac{7\cdot 936\pi}{3}(3^2+2^2+3\times 2)=\dfrac{7\ 936\times\pi\times 19}{3}=158$ cubic in.

Let ABC (Fig 71) be a section through the axis of the cone, then if the length AC be denoted by l, EC is $l-8$ From the similar triangles EFC and ADC,

$$\frac{l}{l-8}=\frac{3}{2}; \qquad l=24 \text{ in } ;$$

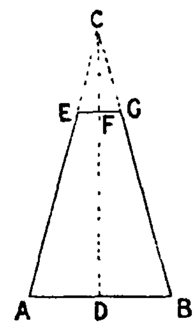

Fig 71

whence the curved surface of the whole cone

$$=\pi\times 3\times 24=72\pi=226\ 2 \text{ sq in}$$

The height CD can be obtained from the right-angled triangle ADC, where $AC=24$ and $AD=3$.

$$\therefore CD=\sqrt{24^2-3^2}=\sqrt{27\times 21}=23\ 81 \text{ in },$$

volume of cone $ABC=\frac{1}{3}\pi\times 3^2\times 23\ 81$

$=224\ 5$ cub in

Having obtained the surface and volume of the cone ABC, it is only necessary to subtract the surface and volume respectively of the smaller cone CEG to obtain the results for the frustum.

As $EC=16$ in ,

lateral surface of cone $CEG=\pi\times 2\times 16=32\pi$;

$\therefore$ surface of frustum $=(72-32)\pi=40\pi$ as before.

Also $CF=23\cdot 81-7\ 936=15\ 87$ in ,

volume of smaller cone $=\dfrac{15\ 87}{3}\times\pi\times 4=66\cdot 5$ sq in. ;

$\therefore$ volume of frustum $=224\ 5-66\cdot 5=158$ cub. in.

EXERCISES. XXIV.

In the following exercises the axis of the solid is assumed to be at right angles to the base unless otherwise expressed

1. Let V denote the volume and S the surface of a pyramid on a square base, given $V=643\ 3$ cub. ft, and the height $h=19\ 36$ ft, find the length of the side of the base and the lateral surface S.

2. The diameter of the base of a cone is 6 inches, altitude 5 inches ; find the volume and curved surface.

3 The volume of a hexagonal pyramid is 249·4 cub. ft ; if the altitude is 8 ft , what is the length of each side of the base?

4 The diameters of the circular ends of the frustum of a lead cone are 4 in and 6 in respectively. The height of the frustum is 3·5 in.; find the volume and the weight. (1 cubic in of lead weighs 0·4121 lbs.)

5. A piece of wood is in the form of a square pyramid; the side of the base is 6 inches, and height 8 in. Find the surface, volume and weight (if the specific gravity of the material be 0 53).

6 The base of a right cone is an ellipse whose axes are 21 ft and 14 ft respectively. The altitude is 12 ft ; find the volume

7. If a right cone on a circular base be divided into three portions by two sections parallel to the base at equal distances from the base and vertex and from one another, compare the three volumes into which it is divided

8 Find the cost of the canvas, 2 ft. wide at 3s. 6d a yard, required to make a conical tent, 12 feet diameter and 8 ft high, taking no account of waste.

9 The base of a pyramid is a triangle whose sides measure 72, 58, and 50 inches ; if the volume is 48 cubic feet, what is the height of the pyramid?

10 What is the volume and the total surface of a frustum of a cone, 42 ft diameter at the base, 21 ft diam at the top, and 14 ft. high

11. The base of a pyramid is an equilateral triangle, length of side 10 inches, height 12 inches. Find the volume

12. Find the curved surface of the frustum of a cone, top and bottom diameters 4 and 6 ft respectively, slant side = 8 ft

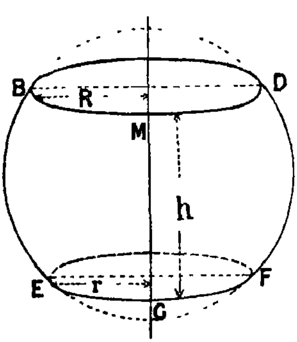

FIG 72 —Zone of a sphere

Sphere.—If S denote the surface, and V the volume of a sphere of radius, r, or diameter d,

$$S=4\pi r^2=\pi d^2,$$

$$V=\tfrac{4}{3}\pi r^3=0\ 5236d^3$$

For proof of these rules see p 411

The ratio of V to S is $\tfrac{1}{3}r$, hence $3V\div S=r$

Zone of a sphere.—Any plane cuts a sphere in a circle Let two parallel planes cut a sphere in two circles BMD, EGF (Fig 72), and let R and r denote the radii of the two circles. The distance

between the planes, usually known as the *thickness of the zone*, may be denoted by h, radius of sphere $= r_1$.

$$S = 2\pi r_1 h, \qquad \text{(i)}$$

$$V = \frac{\pi h}{2}(R^2 + r^2) + \frac{\pi h^3}{6} \qquad \text{(ii)}$$

The result for the convex surface may be stated as follows

Convex surface of zone = **(circumference of a great circle of the sphere)** × **(thickness of zone)**, showing that the surface of a zone depends only on the radius of the sphere and the thickness of the zone Hence, all zones cut from the same, or equal, spheres and having the same thickness, have equal convex surfaces It follows that if a cylinder be circumscribed to a sphere, then, if d_1 denote the diameter,

curved surface of cylinder $= \pi d_1 \times d_1 = \pi d_1^2 =$ surface of sphere.

Segment of a sphere.—As the plane EGF approaches C, the radius r diminishes, and when the plane touches the sphere, r is zero The zone then becomes a segment of a sphere BCD

If S denote the convex surface and h the height of the segment,

$$S = 2\pi r_1 h,$$

the same as in Eq. (i)

The volume may be obtained by putting $r = 0$, in Eq. (ii) and we obtain

$$V = \frac{\pi h}{2} R^2 + \frac{\pi h^3}{6}$$

$$= \frac{\pi h}{6}(3R^2 + h^2). \qquad \text{(iii)}$$

It should be noticed that the surface and volume of a sphere may be obtained from Eq (i) and Eq (ii). Thus, if both the planes touch the sphere, then h, the distance between them, is $2r$, and Eq (i) becomes

$$S = 2\pi r_1 \times 2r_1{}^2 = 4\pi r_1{}^2.$$

Also, when the planes touch the sphere, R and r are both zero. Hence, from Eq. (ii) we obtain,

$$V = \frac{\pi h^3}{6} = \frac{4}{3}\pi r_1{}^3.$$

From (i) we find that to obtain the convex surface of a zone or segment of a sphere it is necessary to ascertain the radius of the sphere

Ex. 1. The diameter of a sphere is 22·48 inches; find its surface and volume Let d denote the diameter

$$S=\pi d^2=\pi\times(22\cdot48)^2;$$

$$\therefore \log S=2\log 22\cdot48+\log\pi=3\cdot2006=\log 1587;$$

$$S=1587 \text{ sq in}$$

$$V=0\cdot5236d^3,$$

$$\log V=\log 0\cdot5236+3\log 22\cdot48=3\cdot7741=\log 5944;$$

$$\therefore V=5944 \text{ cub in}$$

Ex. 2. The inside diameter of a hollow sphere of cast iron is the fraction 0·57 of its outside diameter Find these diameters if the weight is 60 lb Take one cubic inch of cast iron as weighing 0·26 lb.

Let r denote the external radius, then the inside radius will be $0\cdot57r$, and volume of sphere is

$$\frac{4}{3}\pi r^3-\frac{4}{3}\pi(0\cdot57r)^3$$

As 1 cubic inch weighs 0·26 lb., the volume of the sphere is $\frac{6000}{26}$;

$$\therefore \frac{4}{3}\pi r^3\{1-(0\cdot57)^3\}=\frac{6000}{26},$$

$$\therefore \cdot8148r^3=\frac{4500}{26\pi},$$

or
$$r^3=\frac{4500}{26\pi\times0\cdot8148};$$

$$\therefore r=4\cdot074;$$

$$\text{external diameter}=2\times4\cdot074=8\cdot148 \text{ inches},$$

$$\text{internal} \quad ,, \quad =8\cdot148\times0\cdot57=4\cdot644 \text{ inches}.$$

When the outside diameter alone is made 1 per cent smaller, then percentage diminution of weight is

$$\frac{\{1^3-(1-0\cdot01)^3\}}{1-(0\cdot57)^3}\times100=3\cdot6\%$$

Ex. 3. What is the area of the convex surface of the segment of a sphere, the height being 8 inches and diameter of sphere $10\frac{1}{2}$ inches?

$$S=\pi\times10\cdot5\times8$$

$$=263\cdot9 \text{ sq. in}$$

Ex. 4. Find the convex surface and the volume of the zone of a sphere, radii of the two ends 10 inches and 2 inches, and thickness of zone 6 inches.

Let $ABFE$ be the zone and C the centre of the sphere.

Join C to A and E, and draw a line through C perpendicular to AB and EF.

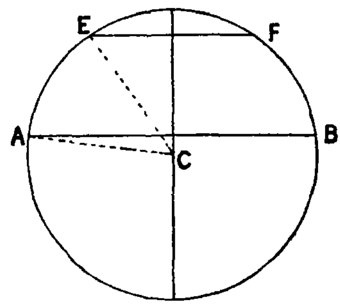

FIG. 73.

If r denote the radius of the sphere, and x the perpendicular distance from C to AB, then

$$r^2 = 10^2 + x^2,$$

and similarly,

$$r^2 = 2^2 + (6 + x)^2.$$

Hence,

$$100 + x^2 = 4 + 36 + 12x + x^2,$$

or $\quad 12x = 60, \quad x = 5;$

$$\therefore r = \sqrt{10^2 + 5^2} = \sqrt{125}$$
$$= 11 \cdot 18 \text{ in}$$

$$\text{Convex surface} = 2\pi \times 11\ 18 \times 6$$
$$= 421\ 5 \text{ sq in}$$

$$\text{Volume of zone} = \frac{6\pi}{2}(10^2 + 2^2) + \frac{\pi \times 6^3}{6}$$
$$= 348\pi \text{ cub. in}$$
$$= 1093 \text{ cub in}$$

EXERCISES. XXV.

1 In a sphere of radius r the surface S and volume V may be obtained from $S = 4\pi r^2$ (i) $V = \frac{4}{3}\pi r^3$ (ii).

(i) Given $r = 6\ 25$ in, find S and V.

(ii) Find r when V is 1 cub. ft.

(iii) Find r when S is 1 sq ft

2. In a spherical zone the height is 4 in, the radii of the two ends being 8 in. and 5 in respectively. Find the convex surface and the volume

3. If the radii of the two circles of a spherical zone are 12·5 in. and 4 25 in. and the thickness of the zone 6 in., what is its volume, its convex surface, and its total surface?

4. The radii of the internal and external surfaces of a hollow spherical shell are 3 ft. and 5 ft. respectively. If the same amount of material were formed into a cube what would be the length of an edge?

5. A cubical box, 5 feet deep, is filled with layers of spherical balls, whose diameters, where they touch, are in vertical and horizontal lines. Find what portion of the space in the box would be left vacant.

6. A circular disc of lead, 3 inches in thickness and 12 inches diameter, is wholly converted into shot of the same density, and of 0·05 inch radius each. How many shot does it make?

7. Find the volume of the segment of a sphere, the radius of the base being 11·83 inches and the radius of the sphere 12 inches.

8. A ball of iron 4 inches diameter is covered with lead. Find the thickness of the lead so that (*a*) the volumes of the iron and lead are equal, (*b*) the surface of the lead is twice that of the iron.

Similar solids.—Two bodies of the same shape are said to be **similar** when the linear dimensions of one are each in proportion to the dimensions of the other. Or, two figures are similar when made to the same drawings but to different scales.

If the linear dimensions of one solid are n times that of another, then the areas of any similar faces are in ratio of n^2 to 1, and the volumes are in the ratio of n^3 to 1.

Thus, if the radius of a sphere is twice those of another, the area, or surface, of the first is 2^2 or 4 times that of the second, and the volume is 2^3 or 8 times that of the second. Thus, if the first weighs 16 lbs., the second will weigh 2 lbs.

Ex. 1. Compare the surfaces of a cube, cylinder, and sphere, the volume in each case being one cubic foot. The altitude of the cylinder is equal to the diameter of its base.

Let a denote, in inches, one side of the cube.

Then $$a=\sqrt[3]{1728}=12,$$

$$S=6a^2=864 \text{ sq. in.}$$

For the cylinder

$$\pi r^2\times 2r=1728\,; \qquad \therefore\ r=\sqrt[3]{\frac{864}{\pi}}.$$

$$\text{Surface of cylinder}=2\pi r(h+r)=2\pi r(2r+r)$$

$$=6\pi r^2=6\pi\times\left(\frac{864}{\pi}\right)^{\frac{2}{3}}$$

$$=797\cdot 3 \text{ sq. in.}$$

For the sphere we have $\frac{4}{3}\pi r_1^3 = 1728$,

$$r_1^3 = \frac{1728 \times 3}{4\pi} = \frac{1296}{\pi} \quad \therefore r_1 = 7\cdot444.$$

$$\text{Surface} = 4\pi r_1^2 = 4\pi\left(\frac{1296}{\pi}\right)^{\frac{2}{3}}$$

$$= 696\cdot5 \text{ sq in}$$

Similarly, if the altitude of a cone is equal to the diameter of the base **and** the volume is one cubic foot, then

$$\text{volume of cone} = \frac{1}{3}\pi r_2^2 \times 2r_2 = 1728\ ;$$

$$r_2^3 = \frac{2592}{\pi}. \qquad r_2 = 9\cdot378.$$

If l denotes length of slant side, then

$$l = \sqrt{2^2r^2 + r^2} = r\sqrt{5}$$

$$\text{Surface of cone} = \pi r(l + r)$$

$$= \pi \times \sqrt[3]{\frac{2592}{\pi}}\left\{\left(\frac{2592}{\pi}\right)^{\frac{1}{3}} \times \sqrt{5} + \left(\frac{2592}{\pi}\right)^{\frac{1}{3}}\right\}$$

$$= 894\cdot1 \text{ square inches}$$

Guldinus' Theorems.—We have already found that surfaces may be generated by the revolution of a line (straight or curved) about an axis, and a solid by the revolution of an area Familiar examples are cylinders, cones, spheres, etc In general, any line, straight or curved, will, when rotating about a given axis, generate a surface called a **surface of revolution.** In like manner an area will generate a **solid of revolution.** The area of the surface, or the volume of the solid, may be obtained by means of two theorems, known as Guldinus' theorems These are as follows

(i) The area of a surface, traced out by the revolution of a curve about an axis in its own plane, is equal to the product of the perimeter of the curve and the distance moved through by its centre of gravity

(ii) The volume, generated by the revolution of such a curve, is the product of the area enclosed by the curve and the distance moved through by the centre of area or centre of gravity.

For proofs of these rules see page 425.

Solid ring.—If a circular disc, whose centre is C, rotates about an axis AD, the solid described is called a **solid circular ring.** The circle C would be the cross section of the ring. Such a ring may be considered as a cylinder bent into a circular form. Familiar examples of solid rings are found in anchor rings, umbrella rings, curtain rings, etc. If r is the radius of cross-section and R the mean radius or length DC,

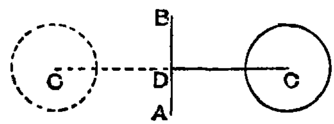

FIG 74

area of ring $= 2\pi r \times 2\pi R$

$= 4\pi^2 Rr,$

***i.e.* curved surface of a ring is equal to the perimeter or circumference of a cross-section multiplied by the circumference of the circle passed through by the centre of gravity of the boundary.**

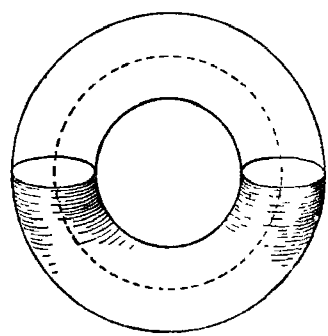
FIG 75 —Solid ring

Volume $= \pi r^2 \times 2\pi R$

$= 2\pi^2 Rr^2,$

***i.e.* volume of a ring is the area of a cross-section multiplied by the circumference of the circle described by the centre of area**

A similar formula may be used when the cross-section of the ring is a rectangle

Cylinder.—If a line CD (Fig 76) rotates about an axis AB, and at a distance r from it, it will trace out the curved surface of a cylinder. The rectangle $ABCD$ will, in a similar manner, trace out the volume of a cylinder

If h denote the distance of CD, then as the centre of gravity of CD is at a distance r from AB, the surface is given by

$$S = h \times 2\pi r = 2\pi rh$$

The area of the rectangle is rh,

Distance moved through by centre of area

$$= 2\pi \times \frac{r}{2} = \pi r;$$

$$V = rh \times \pi r = \pi r^2 h$$

Other cases may be treated in like manner.

A rectangle $ABCD$, when made to rotate about an axis EF parallel to AB, and at a distance r from it, will generate a **hollow cylinder**. Then, if R denote the distance from CD to EF, and h the height of the rectangle, AD will be $R-r$, also distance of centre of area from EF will be $\frac{1}{2}(R+r)$.

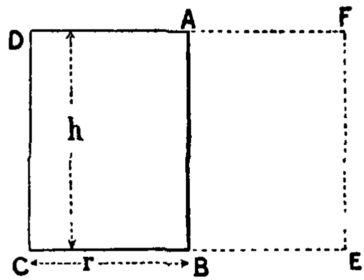

FIG 76

Area of $ABCG=(R-r)h$,

volume

$$=(R-r)h\times\frac{2\pi(R+r)}{2}$$
$$=\pi(R^2-r^2)h.$$

When h is small compared with R, the short cylinder so formed is usually called a flat ring.

Ex 1 *The cross-section of a ring is an ellipse whose principal* diameters are 2 inches and $1\frac{1}{2}$ inches; the middle of this section is at 3 inches from the axis of the ring, what is the volume of the ring?

$$\text{Area of cross-section}=\left(2\times 1\tfrac{1}{2}\right)\frac{\pi}{4}$$

Distance moved through by centre of area in one revolution

$$=2\pi\times 3,$$

$$\therefore\ \text{volume of ring}=\left(2\times 1\tfrac{1}{2}\right)\frac{\pi}{4}\times 2\pi\times 3\,;$$
$$=\frac{9\pi^2}{2}=44{\cdot}43\text{ cub. in}$$

Any irregular area.—In the case of an irregular area, Simpson's Parabolic Rules, the Trapezoidal, Mid-ordinate, or any of the methods usually adopted, may be used to find the area of the figure The position of the centre of area may be found graphically, experimentally, or by calculation Then, the volume traced out can be obtained by application of the rule.

Centre of gravity.—The centre of gravity, or centre of area, of a plane figure may be obtained graphically, experimentally, or by calculation. To obtain accurately the position of the point, it is in many cases necessary to apply the methods

of the Integral Calculus (p 424). In some few cases, however, and especially where the surface is one of revolution, more elementary methods of calculation may be adopted

Suppose that a curve whose length is known, is made to rotate about an axis, lying in the same plane but exterior to the curve Then the distance of the centre of gravity from the axis of rotation may be obtained from Guldinus' Theorem Thus, to ascertain the position of the centre of gravity of the arc of a semicircle

Let ABC (Fig 77) represent a piece of wire in the form of a semicircle If made to rotate about a diameter AB, the surface of a sphere will be traced out.

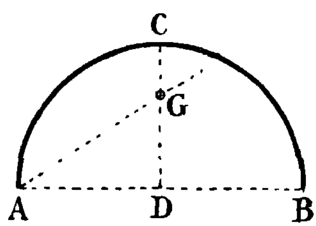

Fig 77.—Centre of gravity of a semicircle

If DC is a line bisecting, and at right angles to, AB, G the position of the centre of gravity, which is from the symmetry of the figure at some point in the line DC, let x denote its distance from AB, and r the radius AD or BD, then the position of G, in terms of r, can be obtained from the first theorem of Guldinus' (p. 217) as follows

Perimeter of curve $=\pi r$

Distance moved through by G is one revolution $=2\pi x$

Surface traced out is the surface of a sphere $=4\pi r^2$,

$$\therefore \pi r \times 2\pi x = 4\pi r^2;$$

$$\therefore x=\frac{2r}{\pi} \qquad \text{(i)}$$

Ex. 1. A piece of wire is bent into the form of a semicircle of 3 feet radius; find the distance of its centre of gravity from the diameter AB.

From (i) $\qquad x=\frac{6}{\pi}=1{\cdot}91$ feet.

In like manner, the centre of gravity of a plane area can be obtained when the volume traced out by it is known. Thus, when it is required to find the centre of area of a semicircle

the volume described is that of a sphere. Let x denote the distance of G from AB.

Then area $= \frac{\pi r^2}{2}$

Distance moved through by $G = 2\pi x$;

$$\therefore \frac{\pi r^2}{2} \times 2\pi x = \tfrac{4}{3}\pi r^3,$$

$$x = \frac{4r}{3\pi} \qquad \ldots\ldots \qquad \text{(ii)}$$

Ex 2 The radius of semicircle is 3 feet, find the distance of its centre of area from the diameter AB

Here, from (ii), we have

$$x = \frac{4}{\pi} = 1{\cdot}274 \text{ feet}$$

Addition and subtraction of solids.—In many cases, to obtain the volume of a solid or a hollow vessel, it may be necessary to add or subtract the volumes of two or more simple solids In other cases a good approximation to the actual volume is obtained by assuming the volume to be represented by that of one or more simple solids, the volume of which can be readily determined

As a simple example, find the weight of water which a tank of the form in Fig 78 can contain The tank is rectangular in plan, its dimensions 6 ft × 4 ft, depth at one end 3 ft., at the other 5 ft.

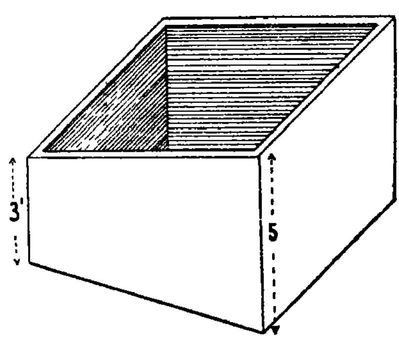

FIG. 78 — Rectangular and triangular prism

The volume is obviously the sum of a rectangular, together with a triangular, prism,

$$\text{volume} = (6 \times 4 \times 3) + \tfrac{1}{2}(2 \times 6 \times 4),$$

$$72 + 24 = 96 \text{ cub ft.};$$

$$\therefore \text{ weight of water} = 96 \times 62{\cdot}3 = 5980{\cdot}8 \text{ lbs}$$

Or, the volume may be obtained as follows

$$\text{Average depth} = \frac{3+5}{2} = 4 \text{ ft.},$$

$$\text{volume} = 6 \times 4 \times 4 = 96 \text{ cub. ft.},$$

$$\text{and weight} = 96 \times 62\cdot3 = 5980\cdot8 \text{ lbs}$$

Cylinder and cone.—An example of a combination of a cylinder and cone is furnished by an ordinary sharpened lead pencil.

Ex 1. A solid consists of a cylinder 6 in. diameter and 3 ft. long, and a cone base 6 in., length 12 in (Fig 79). If one cub in of the material weighs 0·28 lbs., find the weight of the solid

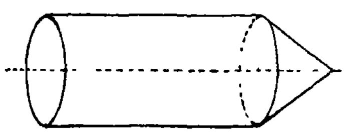

FIG 79 —Cylinder and cone

$$\text{Vol of cylinder} = \pi \times 3^2 \times 36,$$

$$\text{vol of cone} = \frac{\pi \times 3^2 \times 12}{3} = \pi \times 3^2 \times 4\ ;$$

$$\therefore \text{vol. of solid} = \pi \times 9(4+36) = 360\pi \text{ cub in.}$$

$$\text{Weight of solid} = 360\pi \times 0\cdot28 \text{ lbs}$$

$$= 316\cdot7 \text{ lbs}$$

Ex. 2. Find the volume of the solid shown in Fig 80, which consists of the frustum of a cone, 6 ft high, base 6 ft diam., pierced by a cylindrical hole 1 ft diameter, the axis of the cylinder coinciding with the axis of the cone.

The volume is obtained by subtracting the volume of a cylinder from that of the frustum of a cone.

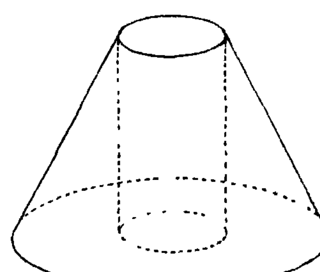

FIG 80 —Frustum of a cone and a cylinder

Volume of frustum

$$= \frac{\pi \times 6}{3}(3^2 + \tfrac{1}{2}^2 + 3 \times \tfrac{1}{2})$$

$$= 2\pi \times 10\cdot75 \text{ cub. ft.}$$

$$\text{Volume of cylinder} = \frac{\pi}{4} \times 1^2 \times 6 = \frac{3\pi}{2} \text{ cub. ft}\ ;$$

$$\therefore \text{volume of solid} = 21\cdot5\pi - 1\cdot5\pi = 20\pi \text{ cub. ft.} = 62\cdot84 \text{ cub ft.}$$

Cylinder and sphere.—When a sphere is pierced by a cylindrical hole we obtain a solid, usually known as a bead. If the axis of the hole is coincident with the axis of the sphere, take the formula for the volume of the zone of a sphere (p. 212), write $R_2 = r_2$, and we obtain

$$\text{volume of zone} = \frac{\pi h}{2}\left(2r_2^2 + \frac{h^2}{3}\right).$$

To obtain the volume of the bead, we must subtract the volume of the cylinder,

$$\text{volume of bead} = \frac{\pi h}{2}\left(2r_2^2 + \frac{h^2}{3}\right) - (\pi r_2^2 \times h)$$

$$= \frac{\pi h^3}{6}.$$

Ex. 3. A cast-iron sphere 12 inches diameter has a cylindrical hole 4 inches diameter bored through it Find the weight of the solid (1 cub. in. weighs 0·26 lb.)

Let x denote the half-height, or thickness OE.

Then $$x = \sqrt{6^2 - 2^2} = \sqrt{32},$$

$$h = 2x = 2\sqrt{32} = 8\sqrt{2} \text{ in.},$$

$$\text{volume of solid} = \frac{\pi(8\sqrt{2})^3}{6} = 758{\cdot}2 \text{ cub. in.},$$

$$\text{weight} = 758{\cdot}2 \times 0{\cdot}26 = 197{\cdot}2 \text{ lbs.}$$

MISCELLANEOUS EXERCISES XXVI.

1 A piece of paper in the form of a circular sector, of which the radius is 7 inches and the curved side 11 inches, is formed into a conical cup Find the area of the conical surface, and also of the base of the cone

2 The interior of a building is in the form of a cylinder of 15 feet radius and 12 feet altitude, surmounted by a cone of equal base and whose vertical angle is a right angle. Find the area of surface and the cubical content of the building

3. What weight of lead weighing 6 lb. per square foot is required to cover a cone 1 ft. in diameter and 2 ft. high? If the covering is to be made with one soldered joint, to what shape should the lead be cut?

4 The slant side of a cone is 25 ft., and the area of its curved surface is 550 sq. ft. Find its volume.

5. Find the lateral surface and volume of the frustum of a cone, slant height of frustum 25 ft. and the diameters of the two ends 5 ft and 27 ft respectively

6. The vertical ends of a hollow trough are equilateral triangles of 12 in side, the bases of the triangles are horizontal, if the length of the trough is 6 ft., find the number of gallons of water it will contain

7. Find the surface of the six equal faces of a hexagonal pyramid, each side of the base being 6 ft, and altitude of pyramid 8 ft.; find also the volume of the pyramid.

8 A cone and a hemisphere have a common base diameter 10 centimetres, find the weight of the solid so formed if the material is steel and the height of the cone is equal to the diameter of the base (1 cubic in steel weighs 0 29 lbs)

9. A cylindrical boiler 4 ft internal diameter and 15 feet long is traversed by 50 tubes, each 3 inches diameter, determine the volume of water the boiler will hold

10 Two thin vessels without lids each contain a cubic foot, the one is a prism on a square base, height equal to half the length of each side of base, the other a cylinder, height equal to radius of base Compare the amounts of material it would require to make them, the thickness being the same for both

11 A pipe supplying 6 gallons of water per minute will fill a hemispherical tank in 4 hours 32 min, find the diameter of the tank

12 Find the volume of a hexagonal room, each side of which is 20 ft and height 30 ft, which also is finished above with a roof in the form of a hexagonal pyramid 15 ft high

13 A lead bar, length 10 cms, width 5 cms, and thickness 4 cms., is melted down and made into 5 equal spherical bullets; find the diameter of each.

14 A sphere of radius r fits closely into the inside of a closed cylindrical box, the height of which is equal to the diameter of the cylinder. Write down the expressions for the volume of the empty space between the sphere and the cylinder If the volume of this empty space is 134 cub in, what is the radius of the sphere?

15 A cast-iron ball of 8 in diameter is coated with a layer of lead 7 in. thick Find the total weight

16. Two spheres of the same material weigh 512 lbs and 729 lbs. respectively, and the cost of gilding the second at 1¾d per sq. in. is £29 13s. 7½d. Find the radius of the first sphere

17. A sphere, whose diameter is one foot is cut out of a cubic foot of lead, and the remainder is melted down into the form of another sphere; find its diameter.

18. A spherical shell of iron, whose diameter is one foot, is filled with lead; find the thickness of the iron, when the weights of the iron and lead are equal (Relative densities are as 1 1·58.)

19. What is the diameter of a sphere which contains 716 cub. in.?

20. The weights of two spheres are as 9:25, and the weights of equal volumes of the substances are as 15 : 9. Compare the diameters.

21. A solid consisting of a right cone standing on a hemisphere is placed in a bath full of water; if the solid is completely immersed, find the weight of water displaced; radius of hemisphere 2 ft., and height of cone 4 ft.

22. The diameters of a spherical shell are 6 in. and 5 in. respectively, and its weight is 13·4 lbs; if the ratio of the weights of equal volumes of lead and iron be as 1·58 to 1, what will be the weight of 12 in length of lead tubing, external diameter 7 in., internal 5 in.

23 Find the radius of a circle whose area is equal to the sum of the areas of two triangles whose sides are 35, 53, 66 ft. and 33, 56, 65 ft.

24. Find the area of the segment of a circle of which the arc is one-third the circumference, the radius being $7\frac{1}{2}$ inches.

25. A piece of copper (specific gravity 8·9) 1 ft. long, 4 inches wide, and $\frac{1}{2}$ inch thick is drawn out into wire of uniform diameter $\frac{1}{16}$ inch. Find the length and the weight of the wire

26. What is the area of a triangle whose sides are 18·40, 13 36, and 15·20 feet?

27. A cubical tank 6 feet edge is half full of water. Find the height to which the surface of the water is raised when an iron cube of 2 ft. edge and an iron sphere 2 ft diameter are placed in the tank

28 A sphere, radius R is pierced by a cylindrical hole whose axis passes through the centre of the sphere If r is the radius of the cylinder, express in terms of r and the radius of the sphere the volume of the bead thus formed If the length of the cylindrical hole be 0 75 in., find the volume of the bead.

29 What is the weight of a cast-iron spherical shell, external diameter 6 in., thickness $\frac{1}{2}$ in?

30. Find the weight of a cast-iron water pipe, 30 inches external diameter, thickness of metal 1 in., length 12 ft.

31. The radii of the two ends of the frustum of a cone are 12 feet and 8 feet respectively; the area of its curved surface is 975·4 square feet. Find the slant height, and volume of the frustum.

32. A frustum of a pyramid has rectangular ends, the sides of the base being 25 and 36 feet; if the area of the top face be 784 sq ft. and the height of the frustum 60 ft, find its volume. Find the radius of a sphere whose volume is equal to the volume of the frustum.

33. Two spheres, each 10 ft. diameter, are melted down and recast into a cone whose height is equal to the radius of its base. Find the height.

CHAPTER XI.

POSITION OF A POINT IN SPACE.

Projections of a line.—To obtain the projections of a line AB on the plane MN (Fig. 81) we may proceed as follows From B and A draw lines Bb, Aa, perpendicular to the plane and meeting the plane in points b and a; then the line joining a to b is the projection required. The angle BHb is the angle between the line and the plane; or if a line AC be drawn through A parallel to ab, then CAB is the angle, θ, of inclination of the line to the plane and

$$ab = AB \cos\theta.$$

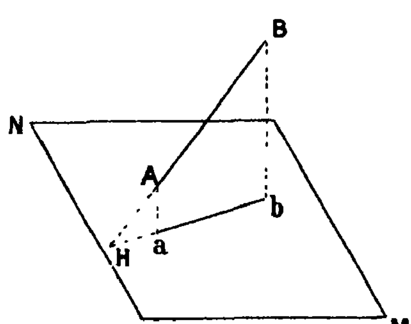

Fig 81 —Angle between a line and a plane

The angle between a line and plane, or the inclination of a line to a plane, is the angle between the line and its projection on the plane. Thus, if BA produced meets the plane NM (Fig. 81) in H, the inclination of the line to the plane is the angle between the line and its projection on the plane. Or, the angle may be obtained by drawing from A a line parallel to ab.

Rabattement.—The graphical method of rabattement is to assume that the line AB rotates about its projection, or plan, ab as an axis until AB lies in the horizontal plane That is, from a and b lines perpendicular to ab and equal in length

to aA, bB, are drawn, and the angle can be measured Such a process is called *rabatting* the line.

Three co-ordinate planes of projections --Very little reflection will convince the student that it is impossible to give measurements which will define the position of a point in space *absolutely* The most that can be done is to choose some point as *origin of co-ordinates*, and take three lines passing through this point (only two of which lie in any one plane) as axes of co-ordinates The three planes which each contain two of these axes are called the co-ordinate planes A point in space may be represented by means of the projections on the three planes; these projections determine the distances of the point from the three planes, and hence the position of the point is known. Usually the planes are chosen mutually at right angles to each other, such as those at one corner of a cube or, roughly, the corner of a room

In the latter case the floor may represent the horizontal plane, sometimes spoken of as the plane xy; one vertical wall the plane xz, and the other vertical wall—at right angles to xz—the plane zy

A model to illustrate these reference planes may be constructed of a piece of flat board (Fig. 82) and two other pieces mutually at right angles to each other It is advisable to have the latter two boards hinged. This arrangement enables the two sides to be rotated until all three planes lie in one plane The planes may be ruled into squares; or squared paper may be fastened on them Then by means of hat pins many problems can be effectively illustrated with the assistance of the model planes

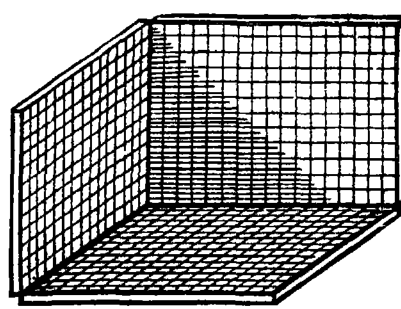

FIG. 82.—Model of the three co-ordinate planes of projection

A model can be more easily made from drawing paper, or cardboard. Draw a square of 9 or 10 inches side (Fig 83).

Along two of its sides mark off distances of 4″ and 6″ and use letters as shown in the illustration. Cut through one of the lines OZ, and fold the paper so that the two points, marked Z, coincide

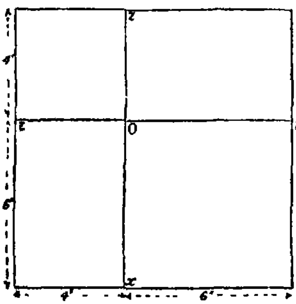

FIG 83

To fix the position of a point in space, imagine such a point P (Fig. 84) From P draw a perpendicular PA to the horizontal plane and meeting it in A AP is the distance of the point P from the plane xy, or, is the z-co-ordinate of P

In a similar manner a perpendicular to the plane yz, meeting it in B, will give the distance of the point from the plane yz; or, the x-co-ordinate of P. Finally, the distance PC, the

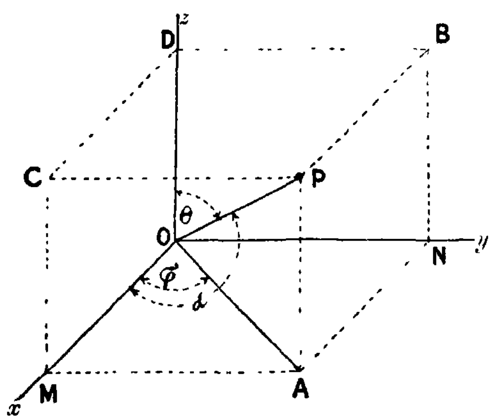

FIG 84

y-co-ordinate of the point, is the distance of the point from the plane zx.

Conversely, given the x-, y-, and z-co-ordinates of a point P, set off $OM=x$, $ON=y$, then the point A is obtained by drawing lines MA, and NA, parallel to the two axes Ox and Oy. AP,

drawn perpendicular to the xy plane equal to z, determines the position of the point P.

From the right-angled triangle, POA (Fig. 84),

$$OA^2 = OM^2 + MA^2 = x^2 + y^2.$$

Also $$OP^2 = OA^2 + AP^2 = x^2 + y^2 + z^2;$$

$$\therefore \quad OP = \sqrt{x^2 + y^2 + z^2}.$$

Thus, the three projections of a point on three intersecting planes definitely determine the distance of a point from these planes.

Negative values of the co-ordinates indicate that the lines affected must be drawn in the opposite direction to that shown in Fig. 84.

It will be found that problems dealing with the projections of a point, line, or plane, may be solved either by graphical methods, using a fairly accurate scale and protractor, or by calculation One method should be used as a check on the other

Ex 1. Given the x-, y-, and z-co-ordinates of a point as $2''$, $1{\cdot}5''$, and $2''$, respectively. Draw the three projections of the line OP on the three planes xy, yz, and zx, and in each case measure the length of the projection Find the distance of P from the origin O, and the angles made by the line OP with the three axes.

Let P (Fig. 84) be the given point and O the origin of co-ordinates. Join OP

The projection on the axis of x is the line OM; on the axis of y is the line ON; and on the axis of z is the line OD

$$OM = 2'', \quad ON = 1{\cdot}5'', \text{ and } OL = 2''.$$

Graphical construction.—The arrangement of the lines and angles can be seen from Fig. 84 To measure the lengths of the lines and the magnitudes of the angles, proceed as follows.

Draw the three axes intersecting at O (Fig. 85), and letter as shown. Set off along the axis of z a distance $OD = 2''$, along the axis of y a distance $ON = 1{\cdot}5''$, and along the axis of x a distance $OM = 2''$. Draw through these points, M and N, lines parallel to the axes to meet in A, and join A to O. Then OA is the projection of OP, on the plane xy its length is $\sqrt{2^2 + (1{\cdot}5)^2} = 2{\cdot}5''$. In a similar manner, the projection OB

on the plane yz, and OC on the plane xz, are obtained; $OB = 2{\cdot}5''$ and $OC = 2{\cdot}83''$.

The distance of P from the origin, or the length of the line OP, is the hypotenuse of a right-angled triangle, of which the base is OA, and the perpendicular AP the height of P above the plane of xy, or simply the z-co-ordinate of the point Hence, as in Fig. 85, draw AP perpendicular to OA and make $AP = OD = 2''$ Join O to P, then $OP = 3\ 2''$ is the distance required

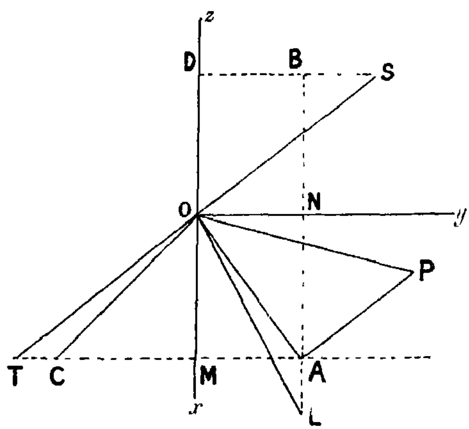

FIG 85

To obtain graphically the angles which the line makes with the three axes it is necessary to **rabat** the line into each of the three planes. Produce NA to L making $NL = OC$. Join O to L. Then the angle NOL is the inclination of the line to the axis of $y = 62°\ 3'$ Similarly, make $DS = OA$ and $MT = OB$ Join S and T to O Then DOS is the angle made by the line with the axis of $z = 51°\ 19'$, and MOT is the angle made by the line with the axis of $x = 51°\ 19'$

A line which passes through two given points may be reduced to the preceding case by taking one of the given points as origin.

Ex. 2. Find the distance between the two points (3, 4, 5 3) (1, 2 5, 3 3) and the angles which the line joining the two given points makes with the axes.

The solution of this problem can be made to depend on the preceding rules by taking as origin the point (1, 2·5, 3 3) The co-ordinates of the remaining points will be (3 - 1) (4 - 2 5) and (5·3 - 3·3) or (2, 1·5, 2) Hence the true length, the projections, and the angles may be obtained as in the preceding example.

The manner in which the three axes are lettered should be noticed. It would appear at first sight to be more convenient to use the horizontal line, drawn from the origin O to the right, as the axis of x instead of y as in the diagram. But when it becomes necessary to apply mathematics to mechanical, or physical, problems, the notation adopted in Fig 84 is more useful, and therefore it is advisable to use it from the commencement.

Calculation.—The preceding results are readily and accurately obtained by calculation

Thus, as in Fig 86, let θ denote the angle which the line OP makes with the axis of z, and ϕ the angle which the projection OA makes with the axis of x Then, the position of P is fixed either when its **Cartesian co-ordinates**, x, y, and z, or its **polar co-ordinates**, r, θ, ϕ, are known; r denoting the length of OP

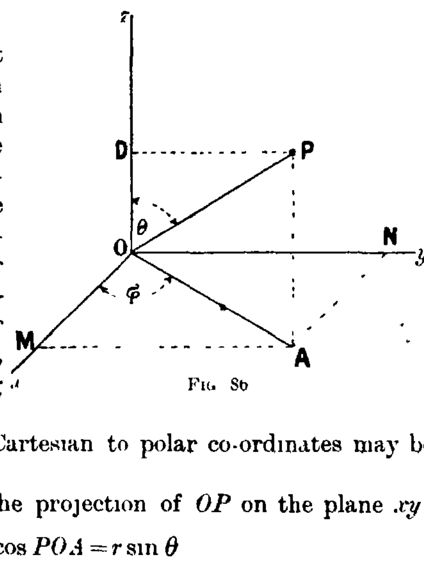

Fig. 86

The conversion from Cartesian to polar co-ordinates may be effected as follows·

From Fig. 86, OA is the projection of OP on the plane xy;

$$OA = OP \cos POA = r \sin\theta$$

Also $\quad OM = x = OA \cos\phi = r \sin\theta \cos\phi$;

$$\therefore \cos\phi = \frac{x}{r \sin\theta} \quad \ldots \quad (i)$$

Or, as $NA = OM$,

$$\tan\phi = \frac{y}{x} \quad \ldots \quad (ii)$$

Thus, ϕ may be found either from (i) or (ii), and when the numerical values of x, y, z, are given, the numerical values of r, θ, and ϕ can be obtained.

Direction-cosines of a line.—As already indicated, when the numerical values of x, y, z, are given, the distance of the point from the origin may be obtained from the relation $r^2=x^2+y^2+z^2$. Hence, we can proceed to find the ratios $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$. These are called the **direction-cosines** of the line.

Thus, if OP (Fig 87) is the line joining the point (x, y, z) to the origin, and α, β, and θ, denote the angles made by the line with the axes of x, y, and z, respectively, then

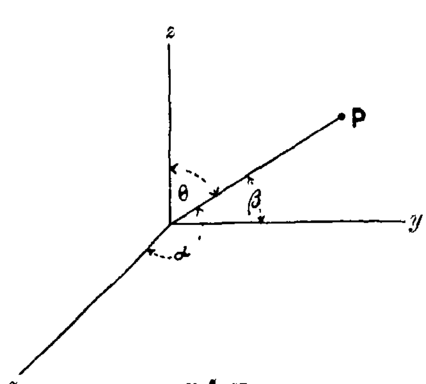

Fig 87

$$\cos\alpha=\frac{x}{OP}=\frac{x}{r},$$

$$\cos\beta=\frac{y}{r},$$

and $$\cos\theta=\frac{z}{r}$$

In this manner the angles made by the line with the three axes can be obtained.

Squaring each ratio and adding,

$$\cos^2\alpha+\cos^2\beta+\cos^2\theta=\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=\frac{x^2+y^2+z^2}{r^2}=1$$

The letter l is often used instead of $\cos\alpha$, and similarly m and n replace $\cos\beta$ and $\cos\theta$ respectively.

From the relation $\cos^2\alpha+\cos^2\beta+\cos^2\theta=1$, or its equivalent, $l^2+m^2+n^2=1$, it will be obvious that, if two of the angles, which a given line OP makes with the axes are known, then the remaining angle can be found. As indicated on page 230 the angles α, β, and θ, can be obtained by construction, but by calculation more accurate results can be obtained.

Ex. 3. *A line makes an angle of 60° with one axis and 45° with another. What angle does it make with the third?*

Let θ denote the required angle.

Then $$\cos^2\theta+\cos^2 60^\circ+\cos^2 45^\circ=1;$$

$$\cos^2\theta=1-\cos^2 60^\circ-\cos^2 45^\circ=\tfrac{1}{4},$$

or $$\cos\theta=\tfrac{1}{2}; \quad \therefore \theta=60^\circ$$

We may repeat Ex. 1 as follows:

Ex. 4. The co-ordinates of a point P are 2, 1·5, 2. Find the distance of the point from the origin, and the angles made by the line OP with the three axes.

$$OP = \sqrt{2^2 + 1{\cdot}5^2 + 2^2} = 3{\cdot}2,$$
$$x = OM = OP\cos\alpha = r\cos\alpha,$$
whence $$\cos\alpha = \frac{x}{r} = \frac{2}{3{\cdot}2} = 0{\cdot}6250, \qquad \alpha = 51^\circ\ 19';$$
$$y = r\cos\beta,$$
or $$\cos\beta = \frac{1{\cdot}5}{3{\cdot}2} = 0{\cdot}4688, \qquad \beta = 62^\circ\ 3';$$
$$z = r\cos\theta,$$
or $$\cos\theta = \frac{2}{3{\cdot}2} = 0{\cdot}6250, \qquad \therefore\ \theta = 51^\circ\ 19'.$$

Ex. 5. If $x=3$, $y=4$, $z=5$, find r, l, m, and n.

$$r^2 = x^2 + y^2 + z^2 = 3^2 + 4^2 + 5^2 = 50,$$
$$r = \sqrt{50} = 7{\cdot}071;$$
$$l = \frac{x}{r} = \frac{3}{7{\cdot}071} = 0{\cdot}4242,$$
$$m = \frac{y}{r} = \frac{4}{7{\cdot}071} = 0{\cdot}5657,$$
$$n = \frac{z}{r} = \frac{5}{7{\cdot}071} = 0{\cdot}7071$$

Ex. 6. The co-ordinates of a point P are (2, 3, 4); find its polar co-ordinates.

$$r = OP = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} = 5{\cdot}385,$$
$$OD = r\cos\theta,$$
$$\cos\theta = \frac{4}{5{\cdot}385} = 0{\cdot}7428, \quad \theta = 42^\circ\ 2';$$
$$x = OA\cos\phi, \text{ and } OA = r\sin\theta,$$
$$\therefore\ x = r\sin\theta\cos\phi, \qquad \text{(i)}$$
or $$y = r\sin\theta\sin\phi, \qquad \text{(ii)}$$

The value of ϕ may be obtained either from (i) or (ii):

Thus $$\sin\phi = \frac{3}{5{\cdot}385 \times \sin 42^\circ\ 2'} = \frac{3}{5{\cdot}385 \times 0{\cdot}6695},$$
$$\log(\sin\phi) = \log 3 - \log 5{\cdot}385 - \log 0{\cdot}6695 = \bar{1}{\cdot}9202,$$
$$\sin\phi = 0{\cdot}8322, \quad \therefore\ \phi = 56^\circ\ 20'.$$

again, dividing (ii) by (i), $\tan\phi = \dfrac{y}{x}$,
$$\tan\phi = \tfrac{3}{2} = 1{\cdot}5, \quad \therefore\ \phi = 56^\circ\ 20'.$$

Angles between a line and the three co-ordinate planes. —Since the angle between a line and a plane is the angle between the line and its projection on the plane, the angle between a line OP (Fig. 84) and the plane xy is the angle between the line and its projection OA on that plane

From the right-angled triangle ONA,

$$OA^2 = ON^2 + NA^2 = y^2 + x^2 ; \quad \therefore \quad OA = \sqrt{x^2 + y^2}$$

Similarly, the projection on the plane $xz = \sqrt{x^2 + z^2}$ and on the plane $yz = \sqrt{y^2 + z^2}$.

Thus, if the three angles made by a line OP with the three co-ordinate planes xy, yz, and zx, be denoted by F, G, and H, respectively, then we have the relations

$$\cos F = \frac{\sqrt{x^2 + y^2}}{r}, \quad \cos G = \frac{\sqrt{y^2 + z^2}}{r}, \quad \cos H = \frac{\sqrt{x^2 + z^2}}{r}.$$

Also $$\cos^2 F + \cos^2 G + \cos^2 H = 2$$

Ex. 7 The three rectangular co-ordinates of a point P are 3, 4, and 2, respectively.

Find

(i) the length of the line OP joining P to the origin O;

(ii) the angles made by the line OP with the three co-ordinate planes xy, yz, and zx;

(iii) the angles which the line OP makes with the three axes

(i) Length $OP = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$
$= 5\ 385$

(ii) The length of the line and the angles may be obtained by graphical methods or by calculation, as follows, F, G, H denoting the angles as above :

The projection of OP on the plane xy is given by $\sqrt{3^2 + 4^2} = 5$.

$$\cos F = \frac{5}{5{\cdot}385} = 0\ 9285 ; \qquad F = 21^\circ\ 48'$$

The projection on the plane zy is

$$\sqrt{4^2 + 2^2} = \sqrt{20} = 4\ 472$$

Let G denote the angle between the line and plane.

$$\therefore \cos G = \frac{4\ 472}{5\ 385} = 0\ 8305 ; \quad \therefore G = 33^\circ\ 52'.$$

The projection on the plane xz is $\sqrt{3^2+2^2}=\sqrt{13}$.

$$\cos H=\frac{\sqrt{13}}{5\cdot385}=0\cdot6696\,; \quad \therefore H=47°\ 58'.$$

(iii) Let α, β, and θ, denote the angles made by the line with the axes of x, y, and z, respectively, then $x=r\cos\alpha$, $y=r\cos\beta$, $z=r\cos\theta$,

$$\cos\alpha=\frac{3}{5\cdot385}=0\cdot5571, \qquad \alpha=56°\ 9',$$

$$\cos\beta=\frac{4}{5\cdot385}=0\cdot7429, \qquad \beta=42°\ 2';$$

$$\cos\theta=\frac{2}{5\cdot385}=0\cdot3714, \qquad \theta=68°\ 12'$$

Ex 8. There is a point P whose x-, y-, and z-co-ordinates are 2, 1·5, and 3 Find its r-, θ-, and ϕ-co-ordinates If O is the origin, find the angles made by OP with the axes of co-ordinates

$$r=\sqrt{x^2+y^2+z^2}=\sqrt{15\cdot25}, \qquad r=3\cdot905\,;$$

$$\tan\phi=\frac{y}{x}=\frac{1\cdot5}{2}=0\cdot75, \qquad \phi=36°\ 52';$$

$$\cos\theta=\frac{z}{r}=\frac{3}{3\cdot905}=0\cdot7683, \qquad \theta=39°\ 48',$$

$$\cos\alpha=\frac{x}{r}=\frac{2}{3\cdot905}=0\cdot5122, \qquad \alpha=59°\ 12';$$

$$\cos\beta=\frac{y}{r}=\frac{1\cdot5}{3\cdot905}=0\cdot3841, \qquad \beta=67°\ 25'.$$

Ex. 9 The polar co-ordinates of a point are $r=5$ feet, $\theta=52°$, and $\phi=70°$, find the x-, y-, and z-co-ordinates. Also find the angles made by the line joining the point to the origin, with the axes of co-ordinates.

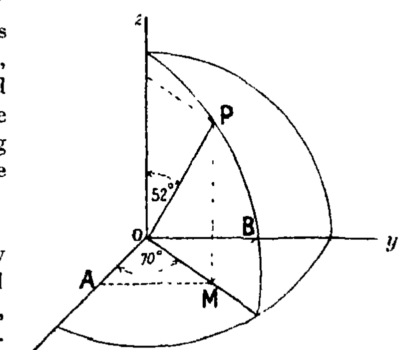

Fig 88

Let P be the given point (Fig 88) Join O to P Then, by projecting on the three axes, OA is the x-co-ordinate; similarly, OB and OC are the y- and z-co-ordinates respectively

$$z=5\cos 52°=5\times0\cdot6157=3\cdot078.$$

$$OM=5\sin 52°=5\times0\cdot7880=3\cdot940.$$

$$x=OM\cos 70°=3\cdot94\times0\cdot342=1\cdot348,$$

$$y=OM\sin 70°=3\cdot94\times0\cdot9397=3\cdot702.$$

Let α, β, and θ, be the three angles made with the three axes.

$$\cos\alpha=\frac{x}{r}=\frac{1.348}{5}=0{\cdot}2696, \quad \therefore \alpha=74^\circ\ 22',$$

$$\cos\beta=\frac{y}{r}=\frac{3.702}{5}=0{\cdot}7404, \quad \therefore \beta=42^\circ\ 14'$$

Line passing through two given points.—If the co-ordinates of two points P and q be denoted by (x, y, z), and (x', y', z'), the equation of the line passing through the two points is

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}.$$

Through P, draw three lines Pp, Pp', Pp'', parallel to the three axes respectively, and draw the remaining sides of the

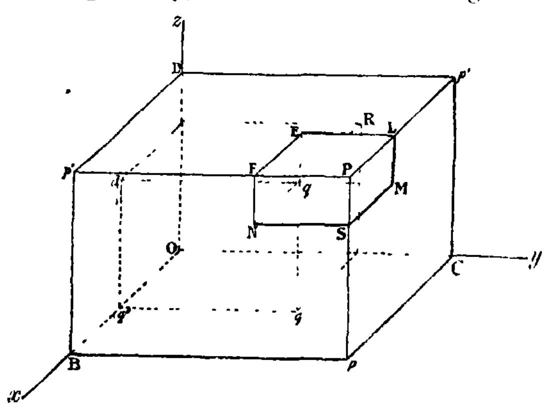

FIG 89 —Line passing through two points

rectangular block as in Fig. 89 Complete a rectangular block having its sides parallel to the former, and q for an angular point.

$$PL=Nq=NR-qR=Pp'-Lp'=x-x',$$
$$PF=Mq=Md-dq=y-y',$$
$$PS=Eq=Eq'-qq'=z-z'$$

Thus, Pq is the diagonal of a rectangular block, the edges of which are $x-x'$, $y-y'$, $z-z'$. Therefore, to find the length of Pq the line joining P and q,

$$Pq=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}.$$

The angle between the line Pq and the axis of z is equal to the angle between Pq and a line qE parallel to the axis of z.

Hence, denoting the angle by θ,

$$n = \cos\theta = \frac{z - z'}{Pq} = \frac{z - z'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}.$$

Similarly, $$l = \frac{x - x'}{Pq}, \quad m = \frac{y - y'}{Pq}$$

When the second point is the origin O, x', y', and z', are each zero, and the equation

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n}$$

becomes $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Ex 10 Find the length of the line joining the two points (7, 9, 11), (3, 4, 5). Find the polar co-ordinates of the line and the angles which the line makes with the three axes of co-ordinates

$$r = \sqrt{(7-3)^2 + (9-4)^2 + (11-5)^2} = \sqrt{77}$$
$$= 8\cdot774;$$

$$z - z' = r\cos\theta, \qquad \cos\theta = \frac{6}{8\cdot774} = 0\cdot6839;$$
$$\theta = 46° \ 51';$$

$$\tan\phi = \frac{y - y'}{x - x'} = \frac{5}{4} = 1\cdot25, \qquad \phi = 51° \ 20';$$

$$\cos\alpha = \frac{x - x'}{r} = \frac{4}{8\cdot774} = 0\cdot4559, \quad \therefore \alpha = 62° \ 53';$$

$$\cos\beta = \frac{y - y'}{r} = \frac{5}{8\cdot774} = 0\cdot5699, \quad \therefore \beta = 55° \ 16'$$

The method is equivalent to shifting the origin to the point (3, 4, 5)

A practical application.—Some of the data we have considered in this chapter may perhaps be better explained by the terms **latitude** and **longitude** of a place on the earth's surface. At regular distances from the two poles a series of

parallel circles are drawn (Fig. 90) and are called *Parallels of Latitude* The parallel of latitude midway between the poles is called the **Equator** These parallels are crossed perpendicularly by circles passing through the poles and called **meridians of longitude.** Selecting one meridian as a standard (the meridian passing through Greenwich), the position of any object *on the earth's surface* can be specified This information, together with the depth below the surface, or the height above it, determines any point or place *on or near* the earth

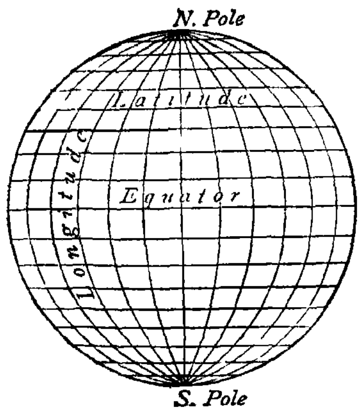

FIG. 90.

The plane xoy may be taken to represent the equatorial plane of the earth, and OZ the earth's axis Then the position of a point P (Fig. 91) on the surface of the earth, or that of a point outside the surface moving with the earth, is known when we are given its distance OP (or r) from the centre, its latitude θ, or co-latitude $(90 - \theta)$, and its ϕ or east longitude, from some standard meridian plane, such as the plane passing through Greenwich

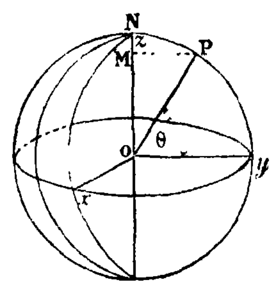

FIG 91

Assuming the earth to be a sphere of radius r, then the distance of a point on the surface can be obtained. If P be a point on the surface, the distance of P from the axis is the distance PM, and

$$PM = r \sin POM = r \cos \theta.$$

***Ex.* 11.** **A point on the earth's surface is in latitude 40° Find its distance from the axis, assuming the earth to be a sphere of 4000 miles radius.**

Required distance $= 4000 \times \cos 40^\circ$

$$= 4000 \times 0{\cdot}766 = 3064 \text{ miles.}$$

Having found the distance PM, the speed at which such a point is moving due to the rotation of the earth can be found.

Ex. 12 Assuming the earth to be a sphere of 4000 miles radius, what is the linear velocity of a place in 40° north latitude? The earth makes one revolution in 23·93 hours

Radius of circle of latitude $= 4000 \times \cos 40^\circ$.

Let s denote the speed

Then
$$s = \frac{4000 \times \cos 40^\circ \times 2\pi}{23{\cdot}93}$$
$$= \frac{4000 \times 0{\cdot}766 \times 2\pi}{23{\cdot}93} = 804{\cdot}4 \text{ miles per hour.}$$

Ex. 13. Find the distance between the two points (3, 4, 5·3) (1, 2·5, 3) and the angles made by the line with the three axes.

Distance $= \sqrt{(3-1)^2 + (4-2{\cdot}5)^2 + (5{\cdot}3-3)^2}$

$$= \sqrt{2^2 + 1{\cdot}5^2 + 2{\cdot}3^2} = 3{\cdot}397$$

$$l = \cos \alpha = \frac{3-1}{3{\cdot}397} = 0{\cdot}5887\,; \qquad \alpha = 53^\circ\ 56'.$$

$$m = \cos \beta = \frac{4-2{\cdot}5}{3{\cdot}397} = 0{\cdot}4416\,; \quad \therefore \beta = 63^\circ\ 48'$$

$$n = \cos \theta = \frac{5{\cdot}3-3}{3{\cdot}397} = 0{\cdot}6770\,; \qquad \theta = 47^\circ\ 24'$$

Cartesian Co-ordinates (two dimensions).—When the given point or points are in the plane of x, y, a resulting simplification occurs. Thus, denoting the co-ordinates of two points P and Q by (x, y) and (a, b), respectively, and the angles made by the line PQ with the axes of x and y by α and β.

Then, if r be the distance between the points,

$$r = \sqrt{(x-a)^2 + (y-b)^2}.$$

Also
$$\frac{x-a}{\cos \alpha} = \frac{y-b}{\cos \beta},$$

$$\therefore\ y - b = \frac{\cos \beta}{\cos \alpha}(x-a)\,;$$

but β is the complement of α;

$$\therefore\ \cos \beta = \sin \alpha.$$

Hence, $y-b=\tan a(x-a)$,

and the equation of the line joining the two points may be written

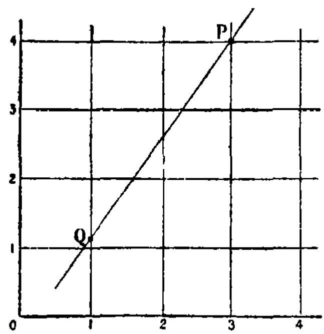

FIG 92

$$y-b=m'(x-a),$$

where m' is the tangent of the angle made by the line with the axis of x.

Thus, given $x=3$, $y=4$, the point P (Fig. 92) is obtained by marking the points of intersection of the lines $x=3$, $y=4$.

In a similar manner, the point Q (1, 1·134) is obtained Join P to Q, then PQ is the line through the points (3, 4), (1, 1 134), and

$$PQ=\sqrt{(3-1)^2+(4-1\cdot134)^2}=3\cdot495,$$

and the equation of the line is

$$y-1\ 134=\frac{2\ 87}{2}(x-1),$$

$$y=1\cdot435x-0\ 3.$$

Polar co-ordinates in two dimensions.—If from a point P a line be drawn to the origin, then if the length of OP be denoted by r, and the angle made by OP with the axis of x be θ, when r and θ are known, the position of the point is determined. Also $x=r\cos\theta$, $y=r\sin\theta$, and the rectangular co-ordinates can be found

Conversely, $r=\sqrt{x^2+y^2}$, $\tan\theta=\frac{y}{r}$

Ex 14. Let $r=20$, $\theta=35°$; find the co-ordinates x and y

Here $x=r\cos 35°=20\times 0\ 8192=16\ 384$;

$y=r\sin 35°=20\times 0\cdot5736=11\cdot472.$

Ex 15. Given the co-ordinates of a point P (4, 3); find the length of the line joining P to the origin and the angle θ.

$$r^2=4^2+3^2=25;\quad\therefore\ r=5;$$

$$\tan\theta=\tfrac{3}{4}=0\cdot75,\quad\theta=36°\ 52'.$$

EXERCISES. XXVII

1. The x- and y-co ordinates of a point A measure 2″ and 3″ and the point is 4″ from the origin. Determine the z-co-ordinate and draw the three projections of A.

2 Obtain the length of the line joining two opposite corners of a rectangular prism 3″ × 5″ × 5″; and find the angles which this line makes with the edges of the solid

3 The co-ordinates of two points P and Q are (3, 1, 2) (4, 2, 4); find the distance PQ

4 The three rectangular co-ordinates of a point P are 3, 4, and 5, determine the polar co-ordinates of the line; the cosines of the angles which the line makes with the three axes

5 The polar co-ordinates of a line joining a point to the origin are $r=3$, $\theta=65°$, $\phi=50°$. Determine its rectangular co-ordinates

6 The co-ordinates of the two points are (3, 4, 5 3) (1, 2 3, 3), find the length of the line joining the two points and the direction-cosines of the line

7. The co-ordinates of two points are (7, 9, 11) and (3, 4 5), find the length of line joining the points and the direction-cosines of the line

8 The polar co-ordinates of a point are $r=5$, $\theta=52°$, $\phi=70°$; find the x-, y-, and z co-ordinates.

9. The co-ordinates of two points A and B are as follows:

Point	x	y	z
A	0 5″	0·8″	3 5″
B	2 4″	3 1″	1 2″

Find the length of the line AB and the cosines of the angles made by the line with the three axes.

10 Given $r=100$, $\theta=25°$, $\phi=70°$, find x, y, z.

11 The three rectangular co-ordinates of a point P are $x=1\ 5$, $y=2\ 3$, $z=1\ 8$ Find the length of the line joining P to the origin and the cosines of the angles which OP makes with the three axes.

12 The polar co-ordinates of a point are $r=20$, $\theta=32°$, $\phi=70°$. Find the rectangular co-ordinates.

13. A point P is 50 inches from the origin, the angles θ and ϕ are 30° and 70° respectively; find the rectangular co ordinates x, y, and z, and the angles made by the line joining P to the origin with the three axes.

In co-ordinate geometry on a plane.

14. Given $r=10$, $\theta=25°$, find x and y

15. Given $x=3''$, $y=4''$, find r and θ.

16 Given $x=5$, $y=8$, find r and θ

17 Given $r=100$, $\theta=15°$, find x and y

18. Given $r=50$, $\theta=20°$, find x and y

CHAPTER XII.

VECTORS.

Scalar quantities.—There are many quantities which can be fully represented by a number. Thus; time, mass, moment of inertia, area, volume, density, temperature, etc., are all examples of so-called **scalar quantities**, or, more shortly, **scalars**, to distinguish them from others called **vectors**, which involve direction as well as magnitude, such as forces, displacements, velocities, accelerations, etc.

In specifying a force, its direction, or sense, and point of application, must be given. The direction may be indicated by using the points of the compass E., W., N., or S., or some intermediate direction. To say that a vector acts in a vertical direction is not sufficiently definite; it must also be stated whether it acts in an upward or a downward direction.

In dealing with vectors in one plane and acting at a point, addition or subtraction may be carried out by calculation or graphically by using a parallelogram or triangle. By resolving a single vector horizontally and vertically two sides of a right-angled triangle are obtained, the hypotenuse giving the sum, or resultant, in magnitude and direction as in Fig. 93.

When the given vectors are all in one plane, but do not act at a point, in addition to the polygon necessary to obtain the magnitude of the resultant, another polygon, called a **funicular** or **link-polygon**, is required to determine its position. In the general case three scalars are necessary. Thus, a vector may be represented in Cartesian co-ordinates by $x-x'$, $y-y'$, $z-z'$ (when one point is the origin this becomes the point x, y, z); or, in polar co-ordinates, by r, θ, ϕ.

Resolution of vectors.—Two vectors acting at a point can be replaced by a single vector which will produce the same effect. Thus, in Fig 93, the two vectors A and B may be replaced by the vector C.

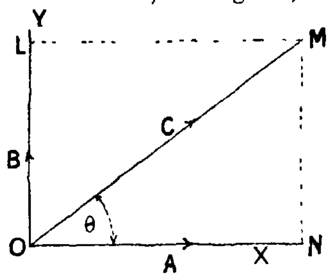

FIG 93 —Resolution of vectors

Conversely, we may replace a single vector by two vectors acting in different directions The two directions usually taken are at right angles to each other

Let OM (Fig 93) represent in direction and magnitude a vector acting at the point O Two lines OX, OY, at right angles to each other are drawn through O. From M, draw MN perpendicular to OX Then ON is the resolved part of the vector C, in the direction OX.

If θ is the inclination of the vector C, then

$$ON = OM \cos \theta.$$

Similarly, if ML be drawn perpendicular to the axis Oy,

$$OL = OM \cos LOM$$
$$= OM \sin \theta.$$

Thus, we obtain two vectors $ON = A$, and $OL = B$, which, acting simultaneously, produce the same effect on the point O as the single vector OM.

This important relation may be stated as follows **The resolved part of a vector in any given direction is equal to the magnitude of the vector multiplied by the cosine of the angle made by the vector with the given direction.**

The two vectors ON and OL are called the **rectangular components** of OM

The process of replacing a vector by its **rectangular components** is called **resolving a vector** The magnitudes of the components may be obtained by drawing the vector C to a convenient scale and measuring the components to the same scale. Or, the magnitudes may be readily obtained by calculation, using either a slide rule or logarithms for the purpose.

Addition of vectors.—Let A and B (Fig. 94) be two vectors. Then, on the two vectors as sides, complete the parallelogram. The diagonal OD denotes the vector sum $A+B$.

Vector subtraction.—What is called vector subtraction may be performed in a manner similar to that adopted in addition, thus, the diagonal lm will represent $A-B$. This may be seen from Fig. 94, in which on is equal to ol, but in the reverse direction; hence, if $ol=B$, $on=-B$. op is the sum of om and on;

$$op=lm=A-B.$$

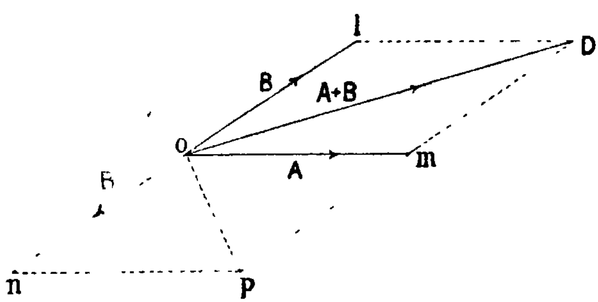

FIG. 94.—Addition of vectors.

In the preceding example $A-B$ may be written $A+(-B)$ or the vector B is added to A after a reversal of direction.

When several vectors act at a point, the sum, or resultant, of the first two can be combined with the third, etc. Or, better, set off a line denoting the magnitude and direction of the first; from the end of this line set off a line equal in magnitude and parallel in direction to the second. Proceeding in this manner, as many different sides of a polygon as there are given vectors are obtained. The magnitude and direction of the line joining the initial position to the final is the resultant in direction and magnitude. A vector equal in magnitude but reversed in sense will balance the given vectors; or, is the **equilibrant** of the given system of vectors. If the lines, drawn in the manner indicated, form a closed polygon, it follows that the given vectors have no resultant; or, in other words, the vector sum is zero. Thus, if the vectors denote displacements,

the effect of carrying out the series of displacements is zero; or, the point having been displaced through the distances indicated by the sides of the polygon is brought back to the starting point. Similarly, if the vectors denote forces, the resultant force is zero, or the given vectors form a system of forces in equilibrium.

Vector equations.—So-called vector equations are for many purposes of the utmost importance, and it is necessary to become familiar with the notation usually adopted to specify a number of vectors acting either in one plane or in various positions in space

Methods which may be adopted in the solution of problems concerning magnitude have already been described, these have been designated as **scalar** We proceed now to extend the idea of equation so as to comprehend the solution of problems concerning **vectors**

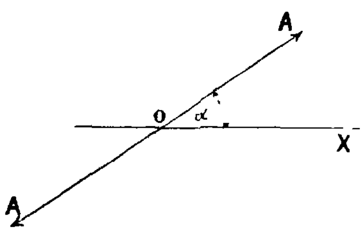

FIG. 95

A relation between a set of vectors is an identity when the result of their actual operation is *nil* Thus, as in Fig. 95, two equal forces acting in the same straight line at an angle α to the line OX may be written as $A_\alpha - A_\alpha = 0$,

or $A_\alpha + A_{180^\circ + \alpha} = 0$.

Similarly, the sum of a vector A in a direction due E, and an equal vector in a direction due W, is zero, or,

$$A_0 + A_{180^\circ} = 0$$

In like manner, the following results follow

$$A_{0^\circ} + A_{120^\circ} + A_{240^\circ} = 0, \quad A_{90^\circ} + A_{180^\circ} + A_{270^\circ} = A_{180^\circ}.$$

The solution of a vector equation is therefore the process of finding a suitable value of R_θ (magnitude and direction). It is necessary to assume an initial line OX from which all angles are measured, the positive direction being anti-clockwise.

In many cases the solution of a given vector equation may be obtained by two or more methods, and one may be used as a check on the other.

Ex. 1 Solve the vector equation

$$R_\theta = A_0 + A_{60^\circ} - A_{240^\circ}.$$

The given vectors may be set out as in Fig 96, in which $oa = A_0$, and ob denotes A_{60°. Also $-A_{240^\circ}$ denotes a vector such as $-bo = ob$, and as this is the same as A_{60°, the given system reduces to

$$R_\theta = A_0 + 2A_{60^\circ}.$$

If the parallelogram $oadb$ be completed on oa and ob as sides, then the resultant, R, is given in magnitude and direction by the diagonal od.

By calculation,

$$(od)^2 = A^2 + (2A)^2 - 4A^2 \cos 120^\circ$$
$$= 7A^2;$$
$$od = A\sqrt{7} = 2{\cdot}645A$$

Let θ denote the angle aod. Then

$$\frac{\sin\theta}{\sin 120^\circ} = \frac{2A}{A\sqrt{7}} = \frac{2}{\sqrt{7}};$$

$$\therefore \sin\theta = \frac{2}{\sqrt{7}} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{21}}{7}$$
$$= 0{\cdot}6546;$$
$$\theta = 40^\circ\ 53'.$$

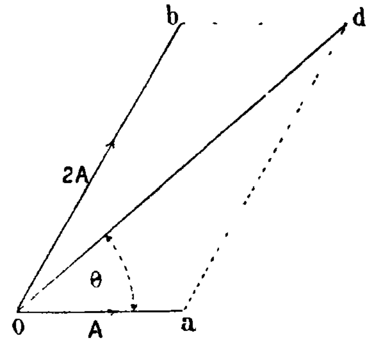

FIG 96.

The result may also be obtained by the process of resolution of vectors, thus:

$$X = A + 2A\cos 60^\circ = 2A,$$
$$Y = 2A \sin 60^\circ = A\sqrt{3},$$
$$R = \sqrt{X^2 + Y^2} = A\sqrt{7} = 2\ 645A,$$
$$\tan\theta = \frac{A\sqrt{3}}{2A} = \frac{\sqrt{3}}{2}; \qquad \theta = 40^\circ\ 53'.$$

As already indicated, when several vectors are given acting at a point, the sum may be obtained by repeated applications of the parallelogram, or better by means of a polygon. Let A, B, C, D (Fig. 97) denote, in magnitude and direction, four vectors acting at a point O. To find the sum we may use the two given vectors as two adjacent sides of a parallelogram, the diagonal of which will give the sum $A+B$. Next, we may use the diagonal and the vector C as two sides of a new parallelogram; and obviously the sum of the given vectors

can be obtained by successive applications But a better method is to form a polygon as follows —From a point a make ab on any convenient scale equal in magnitude and parallel in direction to vector A Similarly, bc is made to represent the vector B, cd to represent vector C, and de to represent vector D Then, the line ae to the same scale denotes the magnitude and direction of the sum of the four given vectors

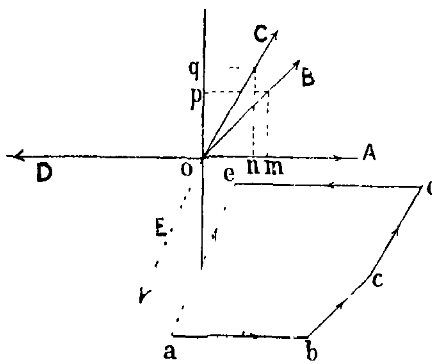

FIG 97

If a vector equal and parallel to ea were to act at O, then the sum of the five vectors

$$A+B+C+D+E$$

would be zero

The sum may also be obtained by resolving the given vectors along and perpendicular to the line OA In this manner two sides of a right-angled triangle are obtained, the hypotenuse of which is the resultant in direction and magnitude Thus, let om and on be the resolved parts of the magnitudes of B and C in the direction OA, then by adding, $OA+om+on-OD$ gives the resolved part of the sum in a horizontal direction, this may be used as the base of a right-angled triangle, the perpendicular being the sum of the distances op and oq. The hypotenuse is the value of R, and the angle θ is the inclination of the hypotenuse to the base

Ex. 2 The magnitudes of four given vectors acting at a point are $A=24$, $B=10$, $C=16$, $D=16$; the angle $AOB=30°$, $AOC=60°$. Find the sum.

If R denotes the sum, and θ its inclination to the horizon, the vector equation may be written

$$R_\theta = 24_{0°} + 10_{30°} + 16_{60°} + 16_{180°}$$

As already described in Fig. 97, make ab equal to 24 on any convenient scale, also bc, cd and de equal to 10, 16 and 16 respectively.

Then R is numerically equal to the length ae, and θ is the angle eab. R is found to be 31, and $\theta = 37^\circ\ 25'$.

The result is also readily obtained by calculation

Sum of horizontal components

$$= 24 + 10\cos 30^\circ + 16\cos 60^\circ + 16\cos 180^\circ$$
$$= 24 + 8{\cdot}66 + 8 - 16 = 24{\cdot}66.$$

Sum of vertical components

$$= 24 \times 0 + 10\sin 30^\circ + 16\sin 60^\circ + 16 \times 0$$
$$= 5 + 13{\cdot}86 = 18{\cdot}86.$$
$$R = \sqrt{(24{\cdot}66)^2 + (18{\cdot}86)^2} = 31{\cdot}04,$$
$$\tan\theta = \frac{18{\cdot}86}{24{\cdot}66} = 0{\cdot}7649,$$
$$\theta = 37^\circ\ 25'$$

Ex 3. Three forces of 27, 52 and 49 lbs. respectively act at a point O; the angle $AOB = 32^\circ$, the angle $AOC = 58^\circ$ Find the resultant in direction or magnitude

The equation may be written in the form

$$R_\theta = A_0 + B_{32} + C_{58^\circ}.$$

Substituting the magnitudes of A, B, and C,

$$R_\theta = 27_{0^\circ} + 52_{32^\circ} + 49_{58^\circ}$$

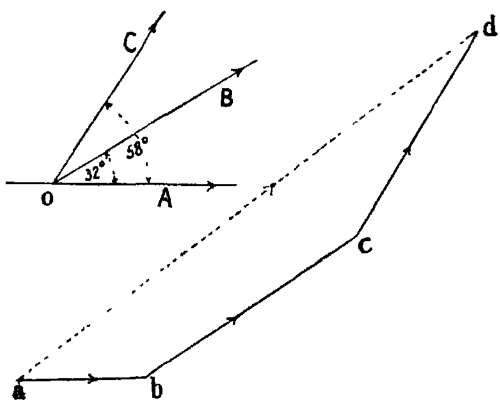

Fig 98

Draw ab (Fig. 98) equal and parallel to vector A, bc equal and parallel to B, and cd equal and parallel to vector C. Then, ad denotes the sum, or resultant, in direction and magnitude.

The magnitude and direction of the resultant may be obtained by calculation. The work may be arranged as follows:

Force	Angle	Horizontal Component	Vertical Component.
27	0°	27	0
52	32°	52 cos 32° = 44·096	52 sin 32° = 27 55
49	58°	49 cos 58° = 25 96	49 sin 58° = 41 55
By addition,		97·06	69 10

$$R=\sqrt{(97\cdot 06)^2+(69\cdot 10)^2}=\sqrt{14196}$$
$$=119\cdot 1,$$
$$\tan\theta=\frac{69\ 10}{97\cdot 06}=0\cdot 7119\ ;$$
$$a=35°\ 27'.$$

One of the most important theorems with regard to vectors is—**that a vector sum is the same in whatever sequence the vectors are added** Thus, if A, B, and C, are three vectors, then it is easy to show either analytically or by graphical construction that $A+B+C=A+C+B$. In fact, the vectors may be added in any convenient manner. This law should be tested in the preceding and the remaining examples.

Ex 4 The magnitudes of four forces acting at a point are 835 400, 650, and 610, and their directions 0°, 58°, 260°, and - 23° (Fig. 99)

Find (i) the direction and magnitude of the line denoting the sum, or resultant, of the forces

(ii) The components resolved along and perpendicular to the initial line.

(iii) The magnitudes of two forces which acting in directions at 70° and 170° will balance the system

(iv) The directions of two balancing forces, magnitudes 500 and 700.

(i) The vector equation may be written

$$R_\theta=835_{0°}+400_{58°}+650_{260°}+610_{-23°}$$

Graphically, make ab on a convenient scale equal to vector A, *i.e.* equal to 835 and horizontal; make bc parallel to vector B

(Fig. 99), and equal to 400. Similarly, cd is made equal and parallel to vector C, and de equal and parallel to vector D. Then, the resultant is the line joining a the initial, to e the final point; the inclination of the line ae to the horizon is the required inclination of the line denoting the sum.

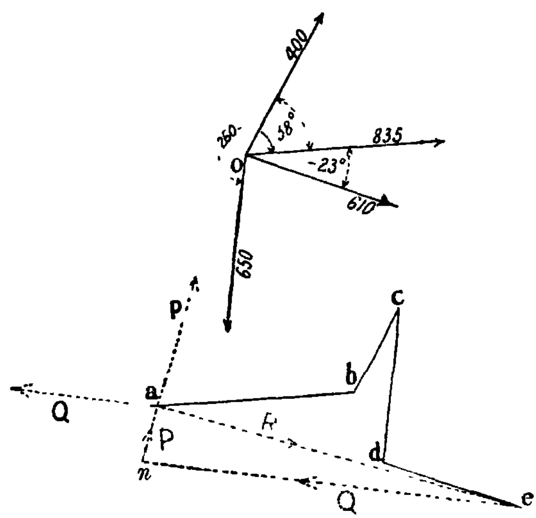

FIG 99.

Or, the sum of the projections on the axes of x and y could be obtained and made to form two sides of a right-angled triangle; the sum of the given vectors is the hypotenuse of the triangle.

Let X denote the sum of the projections on the axis of x

Then, $$\begin{aligned} X &= 835 \cos 0^\circ + 400 \cos 58^\circ - 650 \cos 80^\circ + 610 \cos 23^\circ \\ &= 835 + 400 \times 0{\cdot}5299 - 650 \times 0{\cdot}1736 + 610 \times 0{\cdot}9205 \\ &= 835 + 211{\cdot}96 - 112{\cdot}84 + 561{\cdot}5 \\ &= 1495{\cdot}62. \end{aligned}$$

Similarly, $$\begin{aligned} Y &= 400 \sin 58^\circ - 650 \sin 80^\circ - 610 \sin 23^\circ \\ &= 400 \times 0{\cdot}848 - 650 \times 0{\cdot}9848 - 610 \times 0{\cdot}3907 \\ &= 339{\cdot}2 - 640{\cdot}12 - 238{\cdot}33 \\ &= -539{\cdot}25; \end{aligned}$$

$$\therefore\ R=\sqrt{(1495{\cdot}62)^2+(-539{\cdot}25)^2}=\sqrt{2527679{\cdot}7469}$$
$$=1589{\cdot}87,$$
$$\tan\theta=\frac{Y}{X}=\frac{-539{\cdot}25}{1495{\cdot}62}=-0{\cdot}3602,$$
$$\theta=-19^\circ\ 49'.$$

The work may be arranged as follows.

Force P	Angle	$P\cos a$	$P\sin a$
835 400 650 610	0° 58° 260° −23°	835 211 96 −112 84 561 5	339 2 −640 12 −238 33
		$X=1495\ 62$	$Y=-539\ 25$

Having obtained X and Y, the value of R and θ can be obtained as above

(ii) If X and Y denote the two components at 0° and 90°, then the vector equation may be written

$$X_{0^\circ}+Y_{90^\circ}=835^\circ{}_{0^\circ}+400_{58}+650_{260^\circ}+610_{-23^\circ}$$

The values of X and Y have already been determined, and are 1495 62 and −539 25 respectively

(iii) The inclination of the resultant may be stated as −19° 49′, or 360° − 19° 49′ = 340° 11′ The three forces keeping equilibrium are as indicated in Fig 99 Hence, set off ae equal and parallel to R. Draw a line en parallel to Q, and a line an parallel to P, intersecting the former in n; then, aen is the triangle of forces required, and the magnitudes of P and Q can be measured to the scale on which ae is equal to R.

It will be seen that the triangle aen in Fig 99 which is used to determine the magnitudes of P and Q, could be drawn as a separate diagram.

(iv) The directions are obtained by using a triangle of forces; *i.e* from a and e as centres, and with radii 500, and 700, respectively, describe arcs of circles; then the triangle of forces is obtained, and from this the inclinations may be found.

Some vectors, such as displacements, velocities, accelerations, etc., may be represented by a line, or any parallel line may be used. Such vectors may be called **free vectors**, to distinguish them from other vectors such as **forces**, in which the vectors are localised in a line, and are only free to move in the direction of the length of the line.

Link polygon.—In the preceding example the given vectors have been assumed to act at a point, when this is not the case, it is necessary to obtain the *position* of the resultant, in addition to its magnitude and direction. For this purpose what is called a **funicular** or **link polygon** is used.

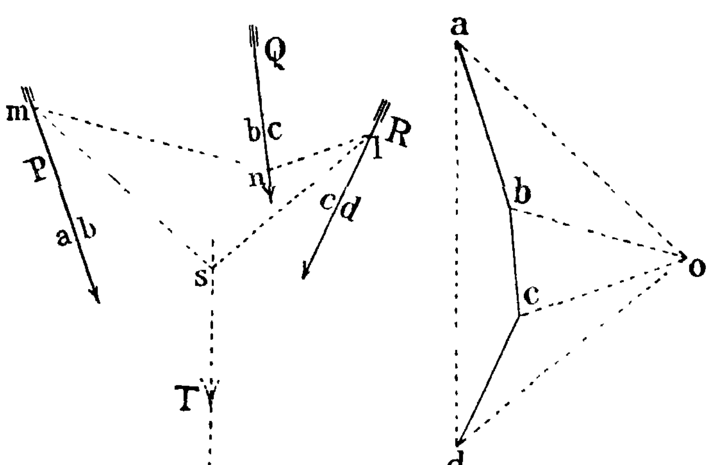

FIG 100 – Vectors which do not act at a point

Given three forces P, Q, and R, which, acting at different points on a rigid body, do not meet at the same point when produced, to find the *resultant* and also its *point of application.*

Instead of denoting a force by a single letter, a very convenient and simple notation is to put a letter on each side of a force, the second letter b for any force P being carried to the first side of the next force Q, thus, in Fig. 100, the force P may be denoted by the letters ab, Q by bc, and R by cd.

In Fig. 100, called the **force polygon,** ab, bc, and cd, are drawn parallel to, and containing as many units of length as there

are units of force in P, Q, R, respectively; the resultant is given in direction and magnitude by the line joining a to d. But this *does not* determine its position. To find the *position* or *point of its application*, we choose any point o and draw radiating lines from o to a, b, c, d

In the space b of the original diagram of forces, at any point m of P draw a line mn parallel to ob intersecting the line of action of the force Q at n In the space c draw a line nl parallel to oc intersecting the force R at l. Finally, draw lines ls, ms parallel to od and oa respectively, intersecting at s. This determines a point on the resultant whose direction and magnitude are indicated by the side ad of the Force Polygon. The whole diagram is now called a Funicular Polygon of the given forces. Evidently the four forces P, Q, R, and T reversed, would, if acting simultaneously, form a system of forces in equilibrium.

Thus, the graphic conditions of equilibrium become

(i) The force polygon must be a closed figure

(ii) The funicular or link polygon must be a closed figure.

Another and very important method which may be used to specify the components and resultant of a given system of forces, is to give, in addition to the magnitude and direction of each vector, the distance from an arbitrary fixed point to the point of intersection of the line denoting a given vector with a horizontal line passing through the point.

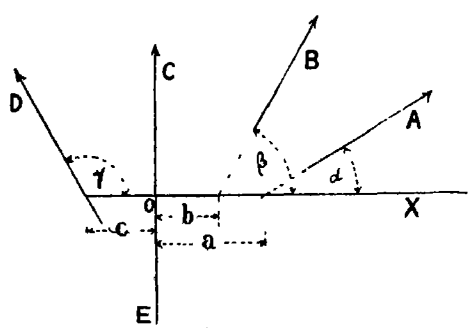

FIG. 101.

This is called the **intercept** of the vector.

Thus, let $ABCDE$ (Fig. 101) be five given vectors in one plane. O is any convenient arbitrary point, and OX a horizontal line passing through O. The distances, a, b, c, of the points, where the lines denoting the vectors intersect the

line OX, are called the *intercepts* of A, B, and D. Thus, the vector A is completely specified by its intercept a, its inclination α, and its sense, indicated by an arrow-head on the line denoting the vector.

In a similar manner, the vector B is specified by its inclination β, its intercept b, and its sense. The vectors C and E pass through the origin and the intercept is zero. In the case of the vector D the intercept is negative or $-d$.

Hence, if R, r, and θ, denote the resultant, its intercept and inclination to OX respectively, then the vector equation may be written

$$_rR_\theta = {}_aA_\alpha + {}_bB_\beta + {}_0C_{90^\circ} + {}_{-c}D_\gamma + {}_0E_{270^\circ}$$

If all the given vectors act at a point the preceding equation becomes: $R_\theta = A_\alpha + B_\beta + C_\gamma + \ldots$

Ex 5. Five vertical forces A, B, C, D, E, are as follows

	A	B	C	D	E
Magnitude in tons,	1·85	3·2	3·2	2 7	3 8
Angle, - - -	270°	90°	270°	270°	90°
Intercept (feet), -	0	4 2	8 2	11·5	16 2

(i) Find the sum of $A+B+C+D+E = {}_rR_\theta$

(ii) „ „ $C+D+E = {}_sS_\phi$.

R is found to be 0 75 tons, $\theta = 270^\circ$, $r = 23$ 6 ft.

The vector equation is

$$_rR_\theta = {}_01\ 85_{270^\circ} + {}_{4\cdot2}3\ 2_{90^\circ} + {}_{8\cdot2}3\ 2_{270^\circ} + {}_{11\cdot5}2\ 7_{270^\circ} + {}_{16\cdot2}3\ 8_{90^\circ}.$$

The given vectors form a system of parallel forces, the sum of the upward components is $3{\cdot}2 + 3{\cdot}8 = 7{\cdot}0$, and of the downward components is $1\ 85 + 3\ 2 + 2\ 7 = 7{\cdot}75$; hence the resultant is $-0{\cdot}75$, and its direction 270°

To find the position of the resultant it is only necessary to take moments about any convenient point such as O Then, if $\bar{x}$ denote the distance of R from O,

$$\bar{x} \times (-0{\cdot}75) = -3{\cdot}2 \times 4{\cdot}2 + 3{\cdot}2 \times 8\ 2 + 2{\cdot}7 \times 11\ 5 - 3{\cdot}8 \times 16{\cdot}2$$
$$= -17{\cdot}71\ ;$$
$$\therefore\ x = \frac{17{\cdot}71}{0{\cdot}75} = 23{\cdot}3.$$

Ex. 6. In the preceding example, if M and N are two points, such that M is -4 and N is 6 feet, respectively, show that the vertical forces, which acting through M and N will balance the given forces, are 2·3 and 1·47, the former at 90°, the latter at 270°.

Ex. 7. Eight gallons of water per second flow through a pipe 6 inches diam. in which there is a right-angled bend; what is the resultant force exerted by the water on the pipe at the bend, neglecting friction?

What is the change in the velocity of the water (that is the *vector* change)? Find the change in the momentum of the water and the resultant force exerted at the bend (1 gallon of water = 0 1605 cub. ft.)

Volume which passes in a second is $8 \times 0{\cdot}1605 \times 1728$ cub in

$$\text{Speed} = \frac{8 \times 0\ 1605 \times 1728}{\pi \times 3^2 \times 12} = 6\ 539 \text{ ft per sec.}$$

Eight gallons $= 10 \times 8 = 80$ lbs.

$$\text{Mass} = \frac{80}{32\ 2}$$

The resultant of two equal forces each equal to $A - A\sqrt{2}$

$$\therefore \text{ change of momentum per sec} = \frac{80 \times \sqrt{2}}{32\ 2} \times 6\ 539 = 22\ 97,$$

$\therefore$ resultant force at bend $= 22{\cdot}97$ lbs

Product of two vectors —The **scalar product** of two vectors **is the product of the two vectors multiplied by the cosine of the angle between them.** The **vector product** may be defined as **the product of the two vectors multiplied by the sine of the angle between them.**

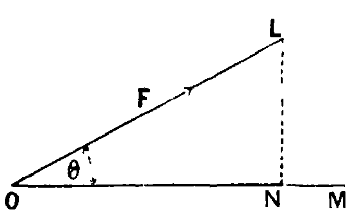

FIG. 102 —Scalar Product

The simplest example of the former occurs in the case of the product of a force and a displacement

If, as in Fig 102, the force F is inclined at an angle θ to the direction in which displacement occurs, then the effective part of F, so far as translation is concerned, is the resolved part of F. Thus, set off OL to represent the force, and draw LN perpendicular to OM; then ON is the resolved part of F in the direction OM, but $ON = OL \cos\theta = F \cos\theta$.

Hence, the product of the two vectors, or work done by the force, is $Fd\cos\theta$, ...(i) where d denotes the displacement

When the angle is 0°, *i.e.* when the direction of the force and the displacement are coincident, since $\cos 0° = 1$, the product is $F \times d$.

When $\theta = 90°$ the force F has no component in the direction of motion, and the work done by F is zero. For any inclination 90° to 180°, the resolved part of F acts in a negative direction, and the work done by F would be in the nature of a resistance or retardation. This would obviously have its maximum value when $\theta = 180°$.

Eq. (i) may be expressed in words as follows

Project one vector on the other, the product of the vector and the projection is the scalar product required. Or, multiply the numerical magnitudes of the two vectors by the cosine of the angle between them.

From Eq. (i) it follows that the product of two unit vectors such as unit force and unit displacement, is $\cos\theta$. In any diagram, when two vectors are shown acting at a point, care must be taken that the arrow-heads denoting the *sense* of each vector are made to go in a direction outwards from the point When this is done, θ is the angle between the vectors.

Ex. 8. The direction of the rails of a tramway is due N., and a force A of 300 lbs in a direction 60° N. of E. acts on the car. Find the work done by the force during a displacement of 100 ft.

If θ denote the angle between the direction of the force A and the direction of the displacement ON, then the resolved part of A in the direction ON is $A\cos\theta$

The product of a force, or the resolved part of a force, and its displacement, or distance moved through, is the work done by the force Thus, in

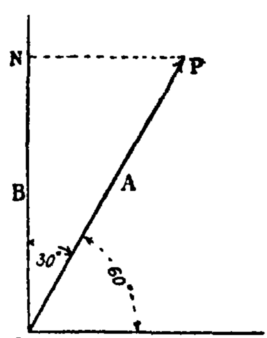

FIG 103.

Fig 103, if B denote the displacement of the car, then the work done is $AB\cos\theta$. ....(i)

As A is 300, $B=100$, and $\theta=30°$,

$$AB\cos 30°=300\times 100\times 0\ 866=25980 \text{ ft.-lbs.}$$

Observe by way of verification that if θ be 0°, then $\cos 0°=1$; the force A is acting in the direction ON, and hence

$$\text{work done}=300\times 100=30,000 \text{ ft.-lbs.}$$

When θ is 90°, then $\cos 90°=0$;

$$\therefore \text{work done}=0.$$

This latter result is obvious from the fact that, when the angle is 90°, the force is in a direction at right angles to the direction of motion, and hence no work is done by the force. Again, if the direction of the force were South, then negative work equal to $-300\times 10=-3000$ ft-lbs. would be done

The vector product is the product of the magnitudes of the two vectors and the sine of the included angle; thus, if θ denote the angle between the two vectors,

$$\text{vector product}=AB\sin\theta. \quad \ldots\ldots \quad (1)$$

If the two vectors are at right angles

$$\sin 90°=1 \text{ and Eq. (1) gives } AB.$$

Vector products are of importance in "couples," etc.

The general case.—In the preceding examples the given vectors have been taken to act in one plane. In the general case, in which the vectors may act in any specified directions in space, the sum or resultant of a number of vectors may be obtained by using, instead of two, the three co-ordinates, x, y, and z The resolved parts of each vector may be obtained, and from these the magnitude and direction of the line representing their sum.

The process may be seen from the following example

Ex. 9. In the following table r denotes the magnitudes of each of three vectors A, B, and C, and α and β the angles made by

Vector.	r	α	β	θ.	x.	y.	z.
A	50	45°	60°	60°	35·35	25	25
B	20	30°	100°	62° 2′	17·31	-3·472	9 59
C	10	120°	45°	60°	-5	7 071	5

each vector with the axes of x and y respectively. Find for each vector the values of θ (where θ denotes the inclination to the axis of z), x, y, and z, and tabulate as shown.

From the given values of α and β the value of θ can be calculated from the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1.$$

Thus, for vector A, we have

$$\cos^2\theta = 1 - \cos^2\alpha - \cos^2\beta = 1 - \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4};$$
$$\cos\theta = \tfrac{1}{2} \text{ and } \theta = 60^\circ.$$

Similarly, for B,

$$\cos^2\theta = 1 - (0{\cdot}866)^2 - (0\ 1736)^2 = 0{\cdot}22; \quad \therefore\ \theta = 62^\circ\ 2'.$$

And, for C,

$$\cos^2\theta = 1 - \tfrac{1}{4} - \tfrac{1}{2} = \tfrac{1}{4}; \qquad \therefore\ \theta = 60^\circ.$$

To obtain the projections x, y, and z of each vector, we use the relations $\quad x = r\cos\alpha, \quad y = r\cos\beta, \quad z = r\cos\theta.$

Thus, for vector A,

$$r = 50^\circ, \quad \alpha = 45^\circ,\ \beta \text{ and } \theta \text{ are each } 60^\circ;$$
$$\therefore\ x = 50\cos 45^\circ = 50 \times 0\ 7071 = 35\ 35,$$
$$y = 50\cos 60^\circ = 50 \times 0{\cdot}50 = 25,$$
$$z = 50\cos 60^\circ = 25$$

For vector B,

$$x = 20\cos 30^\circ = 17\ 32, \quad y = -20\cos 80^\circ = -3{\cdot}472,$$
$$z = 20\cos 62^\circ\ 2' = 9\ 38$$

For C, $\quad x = -10\cos 60^\circ = -5, \quad y = 10\cos 45^\circ = 7{\cdot}071,$

$$z = 10\cos 60^\circ = 5.$$

Adding all the terms in column x and denoting the sum by Σx,

$$\Sigma x = 35\ 35 + 17\ 32 - 5 = 47{\cdot}67.$$

Similarly, $\quad \Sigma y = 25 - 3\ 472 + 7\ 071 = 28\ 6,$

and $\quad \Sigma z = 25 + 9\ 38 + 5 = 39\ 38.$

Hence the resultant of the three vectors is

$$A + B + C = \sqrt{(47{\cdot}67)^2 + (28{\cdot}6)^2 + (39\ 38)^2} - 68{\cdot}1$$

To find the angles made by the resultant vector with the three axes we have

$$\cos\alpha = \frac{47{\cdot}67}{68\ 1} = 0{\cdot}6966; \quad \therefore\ \alpha = 45^\circ\ 35'.$$

$$\cos\beta = \frac{28{\cdot}6}{68{\cdot}1} = 0{\cdot}4181; \quad \therefore\ \beta = 65^\circ\ 10'.$$

$$\cos\theta = \frac{39{\cdot}38}{68\ 1} = 0{\cdot}5788; \quad \therefore\ \theta = 54^\circ\ 38'.$$

Vector algebra.—Many algebraical and trigonometrical relations may be obtained by using vector notation.

Let A and B (Fig. 104) denote two vectors acting at a point O, and let θ denote the angle between A and B

The diagonal of the parallelogram, on the two vectors as sides, is denoted by the sum $A+B$. Let the sides OM, MD, be denoted by a and b respectively, and the diagonal OD by c, and LM by d.

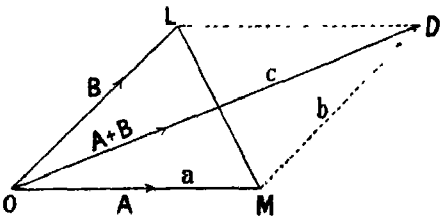

FIG 104

Then $(A+B)^2 = A^2 + 2AB + B^2$.

$A^2 = A \times A$ because the included angle is 0°

Similarly, $B \times B = B^2$.

But, if a and b denote the magnitudes of A and B respectively, then $2AB = 2ab\cos\theta$.

$$\therefore \quad c^2 = (A+B)^2 = A^2 + 2AB + B^2 = a^2 + 2ab\cos\theta + b^2$$

Similarly,

$$\therefore \quad d^2 = (A-B)^2 = A^2 - 2AB + B^2 = a^2 - 2ab\cos\theta + b^2$$

In a similar manner we obtain

$$(A+B)(A-B) = A^2 - B^2, \quad \text{or} \quad cd\cos a = a^2 - b^2\,;$$

$$(A+B)^2 + (A-B)^2 = 2(A^2+B^2), \quad \text{or} \quad c^2 + d^2 = 2(a^2+b^2)\,,$$

$$(A+B)^2 - (A-B)^2 = 4AB, \quad \text{or} \quad c^2 - d^2 = 4ab\cos\theta$$

Again, if the vectors A, B, C represent the sides of a triangle taken in order, $\therefore \; A+B+C=0$.

Let a, b, c, denote the three sides, and α, β, γ, the opposite angles, then,

$$(-A)^2 = (B+C)^2, \quad \text{or} \quad a^2 = b^2 + c^2 - 2bc\cos\theta\,,$$

$$(A+B+C)^2 = 0, \quad \text{or} \quad a^2 + b^2 + c^2 - 2(ab\cos\gamma + bc\cos\alpha + ca\cos\beta) = 0$$

The notation may easily be extended to the case of a plane quadrilateral figure, or a rectangular prism.

Ex. 10. Expand and interpret the following vector equation,

$$D^2=(A+B+C)^2,$$

(*a*) when applied to a plane quadrilateral,

(*b*) when applied to a parallelepiped.

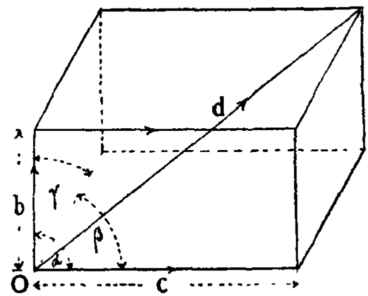

FIG. 105

Let a, b, c, d respectively denote the magnitudes of three edges of a parallelepiped meeting at O (Fig. 105), and α, β, γ signify the internal angles between the sides bc, ca, ab.

In (*a*) we obtain

$$d^2=a^2+b^2+c^2-2ab\cos\gamma-2ac\cos\beta-2bc\cos\alpha,$$

or the square on the diagonal of a quadrilateral is given in terms of the three edges which it meets and their inclination to one another

(*b*) $$d^2=a^2+b^2+c^2+2ab\cos\gamma+2bc\cos\beta+2ac\cos\alpha,$$

or, the square of a diagonal is given in terms of the lengths of the sides and the magnitudes of the included angles.

EXERCISES. XXVIII.

1. The following four forces act in one plane. Determine the resultant, and measure its magnitude, direction and intercept.

	A	B	C	D
Magnitude,	29	18	27	19
Direction,	32°	105°	172°	258°
Intercept,	2·5	1 8	0·5	-0·4

2. The following three vectors A, B, C act at a point; determine the vector sums $A+B+C$ and $A-B+C$, also the direction in each case.

	A	B	C
Magnitude,	37·2	59 5	88·0
Direction,	23°·6	115° 5	238°·0

Verify by construction that $A-(B-C)=A-B+C$. Use a scale of $\frac{1}{2}$ inch to 10 units.

3. Given the following system of coplanar forces, by means of a vector and link polygon determine the resultant of the system. Write down the vector equation.

	A	B	C	D	E
Magnitude,	210	185	313	125	167
Direction,	20°	71°	123°	190°	260°
Intercept,	2·15	1·3	4 6	0	5 5

Find the resultant of A, B and D.

4. Three vectors A, B, C, acting in a horizontal plane, are defined in the following table.

Find the vector sum $A+B+C$; show that $A+B+C=A+C+B$.

	A	B	C
Magnitude,	1 23	1·95	2 60
Direction,	E	33°·2 N. of E	112° N of E

5. A ship A is sailing at 8 7 knots to the east, and a second ship B at 3 4 knots to the south-west. Find the velocity of B relatively to A.

6 Suppose the wind to be blowing at 5 knots from the north. Find the directions which wind vanes would take if carried by the two ships in the preceding exercise.

7. A ship is sailing eastwards at 10 miles an hour. It carries an instrument for recording the apparent velocity of the wind, in both magnitude and direction.

(*a*) If the wind registered by the instrument is apparently one of 20 miles per hour from the north-east, what is the actual wind? Give the answer in miles per hour and degrees north of east of the quarter from which the wind comes.

(*b*) If a wind of 15 miles per hour from the north-east were actually blowing, what apparent wind would the instrument on the vessel register? State this answer in miles per hour and degrees north of east as before.

Use a scale of $\frac{1}{4}$ inch to 1 mile per hour.

8. If three vectors A, B, C are represented by the sides of a triangle taken in order and sense $\therefore (A+B+C)^2=0$ obtain trigonometrical formulae by expanding the following equations:

$$(-A)^2=(B+C)^2, \ (A+B+C)^2.$$

Use a, b, c for the three sides, and α, β, γ for the opposite angles.

9 A ship is sailing at 8·7 knots through water apparently to the east, but there is an ocean current of 3·4 knots to the south-west. Find the actual velocity of the ship as regards the ocean bed.

10. A cyclist rides at 10 miles per hour in a direction due north. Find the apparent direction of the wind which the rider experiences when the actual velocity and direction of the wind is as follows:

(a) 10 miles from E. (b) 10 miles from N.E.
(c) 10 ,, ,, N. (d) 10 ,, ,, N.W.
(e) 10 ,, ,, S.

11. Show that $A_{0^\circ}+A_{120^\circ}+A_{240^\circ}=0$.

12 $A_{90^\circ}+A_{180^\circ}+A_{270^\circ}=A_{180^\circ}$

13 $A_{0^\circ}+A_{60^\circ}+A_{60^\circ}=2{\cdot}645A_{40^\circ\,53'}$.

Solve the vector equations:

14 $R_\theta=10_{0^\circ}-14_{30^\circ}+30_{160^\circ}$. Find R and θ.

15 $A_{60^\circ}+B_{310}+10_{0^\circ}-14_{30^\circ}+30_{160^\circ}=0$. Find A and B.

16. $16_\alpha+25_\beta+10_{0^\circ}-14_{30^\circ}+30_{160^\circ}=0$. Find α and β.

17 $C_{140^\circ}+27_\gamma+10_{0^\circ}-14_{30^\circ}+30_{160^\circ}=0$. Find C and γ.

18. Given the following five vectors:

	A	B	C	D	E
Magnitude, -	20	12	6·8	3·3	15·5
Direction, -	0°	75°	310°	225°	120°

Determine, by constructions, the following vector sums and differences:

(a) $A+B+C+D+E$, (b) $A+B+E+D+C$,
(c) $A+B-C+D-E$, (d) $A+B-E+D-C$.

19. If a vessel steams due N. against a N.E. wind, show in a diagram the direction in which the smoke leaves the funnel.

20. Find A and α in the following vector equation, that is, add the three given vectors, which are all in the plane of the paper.

$$A_\alpha = 3\ 7_{30^\circ} + 1{\cdot}4_{82^\circ} + 2{\cdot}6_{157^\circ}.$$

21. Find B and β from the equation

$$B_\beta = 3\ 7_{30^\circ} - 1{\cdot}4_{82^\circ} + 2{\cdot}6_{157^\circ}.$$

Use a scale 1 inch to 1 unit.

22. Find the resultant or vector sum, that is, find A and α from the vector equation

$$A_\alpha = 26_{35^\circ} + 37_{115^\circ} + 41_{230^\circ}.$$

Use a scale of 1 inch to 10 lbs.

23. Verify by construction that

$$26_{35^\circ} + 37_{115^\circ} + 41_{230^\circ} = 26_{35^\circ} + 41_{230^\circ} + 37_{115^\circ}.$$

24. A mass of 10 lbs has a velocity of $1\ 3_{15^\circ}$ ft per sec It receives a blow which changes its velocity into one of $0{\cdot}8_{100^\circ}$ ft. per sec. What change in the velocity and in the momentum is produced?

25. A point G moves in a straight line. Successive positions of G, measured from a point O in the line at interval of $\frac{1}{40}$ second, are given in the following table

Distance of G (feet), -	0 038	0 302	0 515	0 600	0·515
Time t (seconds), - -	0 0	0 025	0 05	0 075	0 1

Determine successive values of the velocity and acceleration of G. Draw curves showing velocity and time, and acceleration and time. Read off the velocity and acceleration when $t = 0{\cdot}05$ second.

Find R and θ in the following equation:

26. $R_\theta = 20_{0^\circ} + 12_{75^\circ} - 15\ 5_{120^\circ} + 3{\cdot}3_{225^\circ} - 6{\cdot}8_{310^\circ}.$

27. A force acts on a tram-car moving with velocity B. Find $A \times B$ the activity or power in the following cases

	A	B
(a)	300 lbs. E.	20 ft. per sec E.
(b)	250 lbs. N.E.	15 ,, ,, ,,
(c)	200 lbs. N.	20 ,, ,, ,,
(d)	150 lbs. S.W.	10 ,, ,, ,,

28. Solve the vector equation

$$A_{60^\circ} + B_{310^\circ} + 10_{0^\circ} - 15_{30^\circ} + 30_{160^\circ} = 0.$$

29. There are three vectors in a horizontal plane:

A of amount 1·5 towards the south-east.

B of amount 3·9 in the direction towards 20° west of south.

C of amount 2·7 towards the north.

(*a*) Find the vector sums $A+B+C$, (*b*) $A-B+C$, (*c*) $B-C$, (*d*) find the scalar products $A\,.\,B$ and $A\,.\,C$.

30. Values of three vectors acting at a point are given in the following table. Find in each case the value of θ, the magnitudes the angles made with the three axes of the line representing the sum of the three vectors

	r	α	β
A	60	70°	37°
B	50	150°	84°
C	30	85°	170°

CHAPTER XIII.

PROGRESSIONS. BINOMIAL THEORY. ZERO AND INFINITY.

Series.—The term **series** is applied to any expression in which every term is formed according to some common law

Thus, in the series 1, 3, 5, 7 each term is formed by *adding* 2 to the preceding term. In 1, 2, 4, 8 each term is formed by *multiplying* the preceding term by 2

Usually a few terms only are given, these being sufficient to indicate the law which will produce the given terms

The first series is called an **arithmetical progression**, the constant quantity which is added to each term to produce the next is called the **common difference.** The letters A P are usually used to designate such a series

The second series is called a **geometrical progression**, the constant quotient obtained by dividing any term by the preceding term is called the **common ratio** or *constant factor* of the series. The letters G P are used to denote a geometrical progression.

Arithmetical Progression.—A series is said to be an arithmetical progression when any term is formed by adding the same quantity (which may be positive or negative), to the preceding term.

Thus, the series 1, 2, 3, 4 ... is an arithmetical series, the constant difference, obtained by subtracting from any term the preceding term, is unity.

In a series 21, 18, 15, ... the constant difference is -3

Again in a, $a+d$, $a+2d$, ... and a, $a-d$, $a-2d$, ... the first increases and the second diminishes by a common difference d.

In writing such a series, it will be obvious that if a is the first term, $a+d$ the second, $a+2d$ the third, etc., any term

such as the seventh is the first term a together with the addition of d repeated $(7-1)$ times, or is $a+6d$.

If l denotes the last term, and n the number of terms, then

$$l=a+(n-1)d \quad \ldots\ldots\ldots \quad (\mathrm{i})$$

Let S denote the sum of n terms, then

$$S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l.$$

Writing the series in the reverse order we obtain

$$S=l+(l-d)+(l-2d)+\ldots(a+2d)+(a+d)+a.$$

Adding we obtain

$$2S=(a+l)+(a+l)+ \quad \text{to } n \text{ terms}$$
$$=n(a+l),$$
$$\therefore S=\frac{n}{2}(a+l). \quad \ldots\ldots\ldots \quad (\mathrm{ii})$$

From this equation, when a and l are known, the sum of n terms can be obtained.

Again, substituting in (ii) the value of l from Eq. (i), we obtain

$$S=\frac{n}{2}\{2a+(n-1)d\} \quad \ldots\ldots\ldots \quad (\mathrm{iii})$$

Giving the sum of n terms when the first term and the common differences are known

Arithmetical Mean.—If a, A, and b form three quantities in arithmetical progression, then

$$A-a=b-A,$$
$$\therefore A=\frac{a+b}{2};$$

or, *the arithmetical mean of two quantities is one-half their sum.*

Ex. 1 The first term of an arithmetical progression is 3, the third term is 9. What is the sum of 20 terms?

From (i) above, $9=3+2d$;

$$\therefore d=3.$$
$$S=\frac{20}{2}\{6+(20-1)3\}$$
$$=630.$$

Ex. 2. The sum of three numbers in arithmetical progression is 21, and their product is 315. Find the three numbers.

Let $a-d$, a, and $a+d$ denote the three numbers.

$$\therefore\ (a-d)+a+(a+d)=21\ ;$$
$$3a=21,$$
$$a=7.$$

The product of the three terms is

$$a(a^2-d^2)=315\ ;$$
$$\therefore\ 7(7^2-d^2)=315,$$

or
$$49-d^2=45\ ;$$
$$d=\pm 2.$$

Hence, the numbers are 5, 7, 9

Ex. 3 The fifth term of an arithmetical progression is 81, and the second term is 24. Find the series.

$$a+4d=81$$
$$a+\ d=24$$

Subtracting,
$$3d=57\ ;$$
$$d=19 \text{ and } a=5.$$

Hence, the series is 5, 24, 43, .

Ex. 4. Show that if unity be added to the sum of any number of terms of the series 8, 16, 24, etc., the result is the square of an odd number.

$$s=\frac{n}{2}\{16+(n-1)8\}$$
$$=4n^2+4n.$$
$$\therefore\ s+1=4n^2+4n+1=(2n+1)^2,$$

and $(2n+1)^2$ is the square of an odd number.

Ex. 5. Find the sum of the first n natural numbers.

Here $a=1$, $d=1$;

$$\therefore\ s=\frac{n}{2}\{2+(n-1)1\}=\frac{n(n+1)}{2}$$

Sum of squares.—The sum of the squares of the first n natural numbers is often required; if this sum is denoted by Σn^2, then

$$\Sigma n^2=1^2+2^2+3^2+\dots n^2.$$

From the result already obtained (Ex. 5) for the sum of the first n natural number we may infer that the result will contain n^3. In fact, we find

$$n^3-(n-1)^3=n^3-(n^3-3n^2+3n-1)=3n^2-3n+1.$$

As this is true for all values of n, we may write $n-1$ for n, and we obtain

$$(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1.$$

In a similar manner, again writing $n-1$ for n,

$$(n-2)^3-(n-3)^3=3(n-2)^2-3(n-2)+1,$$
$$\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots,$$
$$3^3-2^3=3\times3^2-3\times3+1,$$
$$2^3-1^3=3\times2^2-3\times2+1,$$
$$1^3-0^3=3\times1^2-3\times1+1.$$

By addition we obtain

$$n^3=3(1^2+2^2+3^2+\ldots n^2)-3(1+2+3+\ldots n)+n, \quad\ldots\ldots(1)$$

but $$1+2+3+\ldots n=\frac{n(n+1)}{2};$$

$$\therefore n^3=3\Sigma n^2-\frac{3n(n+1)}{2}+n;$$

or $$3\Sigma n^2=n^3+\frac{3n(n+1)}{2}-n;$$

$$\Sigma n^2=\frac{n(n+1)(2n+1)}{6}.$$

EXERCISES XXIX.

Find the sum of the following series:

1. $4, 3\frac{1}{4}, 2\frac{1}{2}, \ldots$ to 20 terms.
2. $11\frac{2}{3}+10\frac{1}{2}+9\frac{1}{3}+\ldots$ to 21 terms.
3. 14·2, 12·3, 10·4, etc., to 15 terms.
4. $1\frac{2}{3}, 2, 2\frac{1}{3}, \ldots$ to 8 terms.
5. The third term of an A.P. is 7 and the seventh is 3. What is the series?
6. The sum of three numbers in A.P. is 24, and their product is 480. Find the numbers.
7. The sum of n terms of an A.P., whose first two terms are 43, 45, is equal to the sum of $2n$ terms of another progression whose first two terms are 45, 43. Find the value of n.
8. The sum of n terms of the series 3, 6, 9 ... is 975; find n.
9. The sum of 20 terms of an A.P., the first term being 4, is $-62\frac{1}{2}$. Find the common difference.

10. An A.P. consists of 21 terms, the sum of the three last is 117, and of the three middle is 88. Find the first term and common difference.

11. Find the sum of 14 terms of an arithmetical progression whose first term is 11, and common difference 9

12. If the common difference is $-d$, and the sum of n terms $\frac{(2a+d)^2}{9d}$; find n

13. The first term of an A.P. is 5 and the seventh is 23; find the twentieth term.

14 The sum of the first seven terms of an A.P. is 49, the sum of the next eight is 176. Find the series.

15. Find the sum of n terms of the progression

$$(p+1)+(p+3)+(p+5)+$$

If three successive positive terms be taken of any arithmetical progression, show that the ratio of the first to the second is less than the ratio of the second to the third.

16. The first term is 2, the fifth is 18. How many terms must be taken to make the sum 800?

17. The sum of 29 terms is 145, and common difference $\frac{1}{4}$. Find the middle term

18. Find the first term and common difference of an arithmetical progression in which the fifth term from the beginning is 2 and the third from the end -2, the number of terms being 9.

19. If the n^{th} term of an arithmetical series be a given number (A), show that the sum of $2n-1$ terms will be the same, whatever be the first term Find the sum of 7 terms when the 4^{th} term is 11, and verify the preceding statement by writing down and then adding up the seven terms when the first is -4.

Geometrical progression.—A series of terms are said to be in geometrical progression when the quotient obtained by dividing any term by the preceding term is always the same.

The constant quotient is called the **common ratio** of the series.

Let r denote the common ratio and a the first term.

The series of terms a, ar, ar^2, etc., form a geometrical progression, and any term, such as the third, is equal to a multiplied by r raised to the power $(3-1)$ or ar^2.

Thus, if l denote the last term and n the number of terms, then $$l=ar^{n-1}. \quad\text{(i)}$$

Let S denote the sum of n terms, then

$$S = a + ar + ar^2 + \dots ar^{n-2} + ar^{n-1} \quad \dots \quad \dots \quad \dots \text{(ii)}$$

Multiplying every term by r,

$$Sr = ar + ar^2 + ar^3 + \dots ar^{n-1} + ar^n \quad \dots \quad \dots \text{(iii)}$$

Subtract (ii) from (iii).

$$rS - S = ar^n - a,$$

or

$$S(r-1) = a(r^n - 1);$$

$$\therefore S = \frac{a(r^n - 1)}{r - 1} \dots \dots \dots \dots \text{(iv)}$$

Ex 1. The first term of a geometrical progression is 3, and the third term 12 Find the sum of 8 terms.

From (i), $12 = 3r^2$; $\therefore r = \pm 2$.

From (iv), $S = 3\left(\dfrac{2^8 - 1}{2 - 1}\right) = 765.$

Or, using the minus value for r,

$$S = 3\left(\frac{(-2)^8 - 1}{-2 - 1}\right) = -255.$$

Ex. 2. Find the sum of 20 terms of the series

$$3 - 4 + \tfrac{16}{3} - \tfrac{64}{9} + \dots$$

Here $r = -\frac{4}{3}$, $a = 3$, and $n = 20$,

$$S = 3\left\{\frac{(-\frac{4}{3})^{20} - 1}{-\frac{4}{3} - 1}\right\} = -\frac{9}{7}\{(\tfrac{4}{3})^{20} - 1\}.$$

The value of $(\frac{4}{3})^{20}$ is readily obtained by using logarithms.

Sum of an infinite number of terms.—By changing signs in both numerator and denominator, Eq. (iv) above becomes

$$S = \frac{a(1 - r^n)}{1 - r} \dots \quad \dots \quad \dots \quad \dots \text{(v)}$$

When r is a *proper fraction* it is evident that r^n decreases as n increases. Thus, when r is $\frac{1}{10}$, $r^2 = \frac{1}{100}$, $r^3 = \frac{1}{1000}$, etc.; when n is indefinitely great, r^n is zero, and (v) becomes

$$S = \frac{a}{1 - r} \dots \dots \dots \dots \text{(vi)}$$

Hence Eq. (vi) may be used to find the sum of an infinite number of terms; or, as it is called, the sum of a series of terms to infinity.

Ex. 3. Find the sum of the series $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\ldots$ to infinity.

Here $a=\frac{1}{2}$, $r=\frac{2}{3}$;

$$\therefore\ S=\frac{\frac{1}{2}}{1-\frac{2}{3}}=\frac{3}{2}.$$

Ex. 4. Find the sum of the series $0{\cdot}9+0{\cdot}81+\ldots$ to infinity.

Here $a=0{\cdot}9$, $r=0{\cdot}9$;

$$\therefore\ S=\frac{0{\cdot}9}{1-{\cdot}9}=\frac{0{\cdot}9}{0{\cdot}1}=9.$$

Value of a recurring decimal.—The arithmetical rules for finding the value of a recurring decimal depend on the formula for the sum of an infinite series in G.P.

Ex. 5. Find the value of $3{\cdot}\dot{6}$.

$$3{\cdot}\dot{6}=3{\cdot}666\ldots=3+\frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\ldots=3+S;$$

$$\therefore\ r=0{\cdot}1 \text{ and } a=0{\cdot}6;$$

$$\therefore\ S=\frac{0{\cdot}6}{1-0{\cdot}1}=\frac{6}{9}=\frac{2}{3};$$

$$\therefore\ 3{\cdot}\dot{6}=3\tfrac{2}{3}.$$

Geometrical mean.—The positive value of the square root of the product of any two quantities is said to be a *geometric mean* between the other two. The two initial letters G.M. may be used to denote the geometric mean. Thus, if x and y denote two numbers, the A.M. is $\frac{x+y}{2}$, the G.M. is $\sqrt{xy}$.

In the progression 2, 4, 8 .. the middle term 4 is the G.M. of 2 and 8. In like manner in a, ar, ar^2, ar is the G.M. of a and ar^2.

To insert (n − 2) geometric means between two given quantities.

From $$l=ar^{n-1}$$

we obtain $$r^{n-1}=\frac{l}{a},$$

and from this equation when l and a are given r can be obtained.

Ex. 6 Insert four geometric means between 2 and 64.

Including the two given terms the number of terms will be 6, the first term will be 2, and the last 64.

$$r^{6-1} = \frac{64}{2};$$

$$r^5 = 32, \text{ or } r = 2.$$

Hence the means are 4, 8, 16, 32.

Ex. 7. The arithmetical mean of two numbers is 10, and the geometrical mean is 8 Find the numbers.

Let x and y denote the two numbers

Then $$\frac{x+y}{2} = 10; \qquad x+y=20; \quad \ldots \quad (\text{i})$$

$$\sqrt{xy} = 8; \qquad xy = 64 \quad \ldots \quad (\text{ii}).$$

Multiply (ii) by 4 and subtract from (i) squared

$$x^2 - 2xy + y^2 = 144;$$

$$x - y = \pm 12 \quad \ldots \quad (\text{iii})$$

Thus, from (iii) and (i),

$$2x = 32 \text{ or } 8, \qquad x = 16 \text{ or } 4;$$

$$2y = 8 \text{ or } 32; \qquad y = 4 \text{ or } 16$$

Hence the numbers are 16 and 4

MISCELLANEOUS EXERCISES XXX.

Sum the following series

1 $3 + 4\frac{1}{4} + 5\frac{1}{2} +$ to 10 terms

2. $12 + 4 + 1\frac{1}{3} +$ to 10 terms

3. $1{\cdot}48 - 2{\cdot}22 + 3{\cdot}33 -$ to 10 terms

4 $1{\cdot}3 - 3{\cdot}1 + 7{\cdot}5$ to 10 terms

5. $14 + 64 + 114 +$ to 20 terms

6 $14 + 42 + 126$ to 8 terms.

7 $2 + 3\frac{1}{3} + 4\frac{2}{3} +$ to 10 terms

8 $12 + 3 + \frac{3}{4} +$ to 10 terms

9. $0{\cdot}74 - 1{\cdot}11 + 1{\cdot}665 -$

10 $1{\cdot}2 - 2{\cdot}1 + 5{\cdot}4 -$

11 Find the G.P. whose fifth and ninth terms are 1458 and 118098.

12. Find five numbers in G P. such that their sum is 124 and the quotient of the sum of the first and last by the middle term shall be $4\frac{1}{4}$.

13. The continued product of three numbers in G.P. is 64, and the sum of the products of them in pairs is 84. Find the numbers.

14. Sum the series $2\sqrt{2}-2\sqrt{3}+3\sqrt{2}-\ldots$ to 10 terms.

15. Show that $5,\ \frac{5}{3},\ \frac{5}{9},\ \ldots$ to infinity is equal to $3,\ \frac{9}{5},\ \frac{27}{25},\ \ldots$ to infinity.

Sum where possible the following series to infinity:

16 $1,\ -\frac{3}{2},\ +\frac{9}{4}-\ldots$. **17.** $1-\frac{2}{3}+\frac{4}{9}-\ldots$

18. $0{\cdot}9+0{\cdot}81+0{\cdot}729\ldots$. **19.** $56+14+3\frac{1}{2}+\ldots$.

20. $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\ldots$.

21. The fifth term is 81, and the second term 24. Find the series.

22. Find the sum of n terms of the geometrical series

$$1-\frac{3}{2}+\ldots$$

What is the condition that the sum may be negative?

23 The first four terms of a G P. are together equal to 45, and the first six to 189. Find the common ratio and the first term.

24. If the $(p+q)^{\text{th}}$ term of a G.P. be m and the $(p-q)^{\text{th}}$ term be n, show that the p^{th} term is $\sqrt{mn}$.

25. Show that the arithmetic mean between two positive quantities is greater than the geometric mean There is an exceptional case; state it.

Harmonical progression.—A series of terms are said to be in Harmonical Progression when the reciprocals of the terms are in Arithmetical Progression.

Thus, since 1, 2, 3, etc., $\frac{1}{4},\ -\frac{1}{4},\ -\frac{3}{4}$, etc, are in A.P, their reciprocals, $1,\ \frac{1}{2},\ \frac{1}{3}$, etc, and $4,\ -4,\ -\frac{4}{3}$, etc, are in H.P

The preceding rule may be expressed in a more general manner as follows:

Let the three quantities a, b, c be in H.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

As
$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}, \qquad (1)$$

we obtain the relation $a:c=a-b:b-c$, or **three quantities are in H.P. when the ratio of the first to the third is equal to the ratio of the first minus the second, to the second minus the third.**

Again from (i) the harmonical mean between two quantities a and c is $b=\frac{2ac}{a+c}$.

Ex. 1. Find a harmonical mean between 42 and 7.

We may use the formula H M $=\frac{2ac}{a+c}=\frac{2\times 42\times 7}{42+7}=12$, or as $\frac{1}{42}$ and $\frac{1}{7}$ are in A.P ,

$$\text{mean}=\frac{\frac{1}{42}+\frac{1}{7}}{2}=\frac{1}{12}$$

Hence, the required mean is 12, and 42, 12 and 7 are three terms in H.P.

Ex. 2. Insert two harmonical means between 3 and 12

Inverting the given terms we find that $\frac{1}{3}$ and $\frac{1}{12}$ are the first and last terms of an A.P. of four terms; therefore from

$$l=a+(n-1)d$$

we have
$$\tfrac{1}{12}=\tfrac{1}{3}+(4-1)d;$$
$$\therefore 3d=-\tfrac{1}{4}, \text{ or } d=-\tfrac{1}{12}.$$

Hence the common difference is $-\frac{1}{12}$; therefore the terms are

$$\tfrac{1}{3}-\tfrac{1}{12}=\tfrac{1}{4} \text{ and } \tfrac{1}{3}-\tfrac{2}{12}=\tfrac{1}{6},$$

or the arithmetical means are $\frac{1}{4}$ and $\frac{1}{6}$.

Hence the harmonic means are 4 and 6

Let A, G, and H denote the arithmetical, geometrical, and harmonical means respectively between two quantities a and c.

Then
$$A=\frac{a+c}{2},\quad G=\sqrt{ac},\quad H=\frac{2ac}{a+c}.$$

EXERCISES. XXXI.

1. Define harmonic progression; insert 4 harmonic means between 2 and 12

2. Find the arithmetic, geometric, and harmonic means between 2 and 8.

3. Find a third term to 42 and 12 in H.P

4. Find a first term to 8 and 20 in H P.

5. The sum of three terms in H.P. is $1\frac{1}{12}$; if the first term is $\frac{1}{2}$, what is the series?

6. The arithmetical mean between two numbers exceeds the geometric by 2, and the geometrical exceeds the harmonical by 1·6. Find the numbers.

7. A H.P. consists of six terms; the last three terms are 2, 3 and 6; find the first three.

8. Find in H P the fourth term to 6, 8 and 12.

9. Insert three harmonic means between 2 and 3

10. Find the arithmetic, geometric, and harmonic means between 2 and $\frac{9}{2}$, and write down three terms of each series

11. If x, y, z be the p^{th}, q^{th} and r^{th} terms of a H.P., show that

$$(r-q)yz+(p-r)xz+(q-p)xy=0$$

Miscellaneous Series.—The preceding methods may sometimes be adopted to obtain the summation of given series neither in A P nor in G.P. The processes employed may be seen from the following examples

Ex 1 (*a*) Find the sum of the series

$$a+(a+b)x+(a+2b)x^2+ \quad +\{a+(n-1)b\}x^{n-1}$$

(*b*) Show that the sum of the first n even numbers is equal to $\left(1+\frac{1}{n}\right)$ times the sum of the first n odd numbers

(*a*) Let $S=a+(a+b)x+(a+2b)x^2+\ldots\{a+(n-1)b\}x^{n-1}$

Multiplying all through by x,

$$Sx=ax+(a+b)x^2+ \quad \{a+(n-2)b\}x^{n-1}+\{a+(n-1)b\}x^n$$

By subtraction,

$$S(1-x)=a+b(x+x^2+ \quad x^{n-1})-\{a+(n-1)b\}x^n$$

$$=a+\frac{bx(1-x^{n-1})}{1-x}-\{a+(n-1)b\}x^n,$$

or $$S=\frac{a(1-x^n)}{1-x}+\frac{bx(1-x^{n-1})}{(1-x)^2}-\frac{(n-1)bx^n}{1-x}$$

(*b*) The sum of the first n even numbers is an A P Common difference and first term 2

$$S=2+4+6 \quad +2n$$

$$=\frac{n}{2}(2+2n)=n(n+1) \qquad \text{(i)}$$

Similarly, for the sum of the first n odd numbers,

$$S'=1+3+5 \quad +(2n-1)$$

$$=\frac{n}{2}(2n-1+1)=n^2. \qquad \text{(ii)}$$

Hence, comparing (i) and (ii),

$$n(n+1)=n^2\left(1+\frac{1}{n}\right);$$

sum of first n even numbers $=\left(1+\frac{1}{n}\right)$ times the sum of the first n odd numbers.

MISCELLANEOUS EXERCISES. XXXII.

Sum to infinity the following series in G.P.

1. $9{\cdot}6,\ 7{\cdot}2,\ 5{\cdot}4$, etc.

2. $14{\cdot}8,\ 10{\cdot}8,\ 8{\cdot}1$, etc.

3. $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}$, etc.

4. $4-3+\frac{9}{4}$, etc.

5. $84+14+2\frac{1}{3}$, etc.

6. $56+14+3\frac{1}{2}$, etc

7. $0{\cdot}8-0{\cdot}64+$ etc.

8. $7-\frac{7}{4}+\frac{7}{4}$, etc.

9. $2+\frac{8}{7}+\frac{32}{49}$, etc

10. What is meant by the sum of a geometrical series to infinity? Given that r is positive and that

$$(1+r+r^2+\ldots \text{ to infinity})(1+p+p^2+\ldots \text{ to infinity})$$
$$=1+rp+r^2p^2+\ldots,$$

show that p must be negative and r less than $\frac{1}{3}$.

11. The first and second terms of a progression are $5\frac{1}{3}$ and $2\frac{1}{3}$. Find the 4th term on the supposition that the progression is (a) A.P., (b) G.P., (c) H.P.

12. Find the A.P. in which the tenth term is -100 and forty-eighth term is 128.

13. Find the sum of 8 terms of the series $1\frac{2}{3},\ 2,\ 2\frac{1}{3},\ \ldots$, and the sum of 17 terms of $-1\frac{1}{3},\ -1,\ -\frac{2}{3},\ \ldots$.

14. Insert 8 arithmetical means between -250 and 1370. If one arithmetical mean, A, and two geometrical means, p and q, be inserted between two given quantities, show that $p^3+q^3=2Apq$.

15. Insert 8 geometrical means between 512 and 19683. If one geometrical mean, G, and two arithmetical means, p and q, be inserted between two given quantities, show that $G^2=(2p-q)(2q-p)$.

16. Find the sum of n terms of the series 8, 16, 24, ... and show that if unity be added to the sum the result is the square of an odd number.

17. Find the sum of the series

(a) $1+x+x^2+x^3+\ldots+x^{n-1}$,

(b) $1+2x+4x^2+8x^3+\ldots+2^{n-1}x^{n-1}+2^nx^n$,

(c) $1+2x+3x^2+4x^3+\ldots+nx^{n-1}$.

18. If a and b are any two numbers, and A, G, H three other numbers, such that a, b, A are in arithmetical progression, a, b, G in geometrical progression, and a, b, H in harmonical progression, show that $4H(A-G)(G-H)=G(A-H)^2$

19. Find the sum of y^2+2b, y^4+4b, y^6+6b, etc , to n terms.

Binomial Theorem.—By the binomial theorem—one of the most useful theorems in mathematics—any binomial expression, *i.e.* an expression consisting of two terms, can be raised to any required power. The theorem may be stated as follows:

$$(\mathbf{a}+\mathbf{b})^n=\mathbf{a}^n+\mathbf{n}\mathbf{a}^{n-1}\mathbf{b}+\frac{\mathbf{n}(\mathbf{n}-1)}{2}\mathbf{a}^{n-2}\mathbf{b}^2+\frac{\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)}{2\times 3}\mathbf{a}^{n-3}\mathbf{b}^3+\dots \quad (1)$$

The terms on the right-hand side of the equation form what is called the expansion of $(a+b)^n$.

The series on the right terminates only when n is a positive whole number. Thus, when n is 2,

$$(a+b)^2=a^2+2ab+\frac{2\times 1}{2}b^2$$
$$=a^2+2ab+b^2.$$

When n is 3,

$$(a+b)^3=a^3+3a^2b+\frac{3\times 2}{2}ab^2+\frac{3\times 2\times 1}{2\times 3}b^3$$
$$=a^3+3a^2b+3ab^2+b^3.$$

The expansions of $(a+b)^4$, $(a+b)^5$, etc., can be obtained in like manner. In each of the preceding results, where n is a positive integer, the following rules hold

(1) The index of the highest power is n and its coefficient is 1.

(2) Indices of a *decrease* by 1 in each succeeding term, whilst those of b *increase* by one in each term.

(3) Number of terms is equal to index $+1$.

(4) The coefficients of the terms equally distant from the beginning and the end of the series are the same.

When the preceding rules have been carefully studied it will be possible for the student to write down the second, third, or any other term, such as the rth or $r+1$th term, of an expansion. The general or $(r+1)$th term is

$$\frac{\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)\ldots(\mathbf{n}-\mathbf{r}+1)}{\underline{|\mathbf{r}}}\mathbf{a}^{n-r}\mathbf{b}^r,$$

where $\lfloor r$, which is also written $r\,!$, signifies

$$1 \times 2 \times 3 \times \ldots \times r.$$

Note that when $r=0$ the value of $\lfloor r$ is called $=1$.

When n is negative, the series indicated by (1) becomes

$$(\mathbf{a}+\mathbf{b})^{-n}=\mathbf{a}^{-n}-\mathbf{n}\mathbf{a}^{-n-1}\mathbf{b}+\frac{\mathbf{n}(\mathbf{n}+1)}{\lfloor 2}\mathbf{a}^{-n-2}\mathbf{b}^2-, \text{ etc.,}$$

and the general, or $(r+1)^{\text{th}}$, term will be

$$(-1)^r\frac{\mathbf{n}(\mathbf{n}+1)\ldots(\mathbf{n}+\mathbf{r}-1)}{\lfloor \mathbf{r}}\mathbf{a}^{-n-r}\mathbf{b}^r.$$

Ex. 1. Find the 9th term of $(a+b)^{11}$.

Here $\qquad r+1=9\,; \quad \therefore r=8\,;$

$$\therefore \text{ required term}=\frac{11\cdot 10\ldots(11-8+1)a^3b^8}{\lfloor 8}$$

$$=165a^3b^8.$$

The theorem may be applied to expand an expression of more than two terms, thus

$$(a+b+c)^4=(a+b)^4+4(a+b)^3c+6(a+b)^2c^2+4(a+b)c^3+c^4,$$

and each binomial may be expanded in the usual manner.

As n may be an integer, positive or negative, or a fractional number, it follows that a binomial expression may be raised to a given power, or the root of a given number may be obtained from the expansion.

It can be proved that no limit need ordinarily be placed on the value of n. Care should be taken to ensure that an expression which is to be raised to a given power has its greatest term in the first place, especially when n is not a positive integer. When this is done any numerical result can be obtained to any desired degree of accuracy by increasing the number of terms.

Ex. 2 $\quad (17)^{\frac{1}{2}}=(4^2+1)^{\frac{1}{2}}=(4^2)^{\frac{1}{2}}(1+\frac{1}{16})^{\frac{1}{2}}$

$$=4\left\{1+\frac{1}{32}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{\lfloor 2}(\tfrac{1}{16})^2+, \text{ etc.}\right\}$$

$$=4(1+\tfrac{1}{32}-\tfrac{1}{2024}+\text{etc.})=4\ 1228 \text{ approx.}$$

Ex. 3. Find the value of $0{\cdot}9^{\frac{4}{5}}$ by the binomial theorem Compare the result with that obtained by using logarithms.

$$\left(\tfrac{9}{10}\right)^{\frac{4}{5}} = \left(1 - \tfrac{1}{10}\right)^{\frac{4}{5}}.$$

Expanding by the binomial theorem, this becomes

$$1 - \tfrac{4}{5}\left(\tfrac{1}{10}\right) + \frac{\frac{4}{5}\left(\frac{4}{5}-1\right)}{\underline{|2}}\left(\tfrac{1}{10}\right)^2 - \frac{\frac{4}{5}\left(\frac{4}{5}-1\right)\left(\frac{4}{5}-2\right)}{\underline{|3}}\left(\tfrac{1}{10}\right)^3 + $$

$$= 1 - 0{\cdot}08 + \frac{4(-1)2^2}{2 \times 10^4} - \frac{4(-1)(-6)2^3}{\underline{|3} \times 10^6} + \frac{4(-1)(-6)(-11)2^4}{\underline{|4} \times 10^8} - .$$

$$= 1 - 0\ 08 - 0\ 0008 - 0{\cdot}000032 - \frac{11 \times 16}{10^8} +$$

$$= 1 - 0{\cdot}08083376 = 0{\cdot}91916624$$

Using four-figure logarithms we have

$$\log (0{\cdot}9)^{\frac{4}{5}} = \tfrac{4}{5} \log 0{\cdot}9 = \tfrac{4}{5} \times \bar{1}\ 9542 = \bar{1}\ 9633$$

$$= 0\ 9189$$

Approximations.—The binomial theorem gives the expansion of $(1+a)^n$ thus

$$(1+a)^n = 1 + na + \frac{n(n-1)}{1 \times 2}a^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}a^3 + \ .$$

When a is small compared with 1, then a^2 will be very small, and the first two terms of the expansion are sufficiently accurate for many practical purposes Thus

$$(\mathbf{1} + \mathbf{a})^n = \mathbf{1} + \mathbf{na},$$

when a is small compared with 1.

Ex. 4. Find the value of $1\ 05^3$.

$$1\ 05^3 = (1 + 0\ 05)^3 = 1 + 3 \times 0\ 05 + 3(0\ 05)^2 + (0\ 05)^3$$

$$= 1 + 0\ 15 + 0\ 0075 \text{ approx}$$

Using only the first two terms,

$$1\ 05^3 = 1\ 15.$$

It will be noticed that the error introduced only affects the third decimal place, and the numerical magnitude of the error decreases as the term a diminishes.

Again, if $a = 0{\cdot}005$, then

$$1{\cdot}005^3 = (1 + 0\ 005)^3 = 1 + 3 \times 0{\cdot}005 + \frac{3 \times 2}{2}(0{\cdot}005)^2 + (0\ 005)^3$$

$$= 1 + 0{\cdot}015 + 0{\cdot}000075 = 1\ 015075 \text{ approx}$$

The first two terms give $(1{\cdot}005)^3 = 1 + 0{\cdot}005 \times 3 = 1\ 015$, which is quite accurate enough for most practical purposes

We may, of course, use the same rule when n is fractional.

Ex. 5. $\sqrt[3]{1\cdot05}=(1+0\cdot05)^{\frac{1}{3}}$

$$=1+\tfrac{1}{3}\times0\cdot05=1\cdot0167.$$

Ex. 6. $\dfrac{1}{\sqrt[3]{1\cdot05}}=(1+0\cdot05)^{-\frac{1}{3}}$

$$=1-\tfrac{1}{3}\times0\cdot05=1-0\cdot0167=0\cdot9833.$$

Ex. 7. Find the superficial and cubical expansion of iron, taking a, the coefficient of linear expansion, as 0·000012 or $1\cdot2\times10^{-5}$.

If the side of a square be of unit length, then when the temperature is increased by 1° C., the length of each side becomes $1+a$, and the area of the square is $(1+a)^2=1+2a+a^2$;

$$(1+a)^2=1+2\times0\cdot000012+(0\cdot000012)^2.$$

Subtracting the value of the original area from this, we find the coefficient of superficial expansion to be $2\times0\cdot000012+(0\cdot000012)^2$.

As a is a small quantity its square will be very small, even if an exact determination of it were made it would have no appreciable effect on the larger quantity;

coefficient of superficial expansion is $2a=0\cdot000024$ or $2\cdot4\times10^{-5}$.

In a similar manner $(1+a)^3$ (when a is a small quantity compared with unity) may be written as $1+3a$;

coefficient of cubical expansion for the same material

$$=3a=0\cdot000036=3\cdot6\times10^{-5}.$$

Again, by multiplication,

$$(1+a)(1+b)=1+a+b+ab,$$

when a and b are both small compared with unity, we may write

$$\mathbf{(1+a)(1+b)=1+a+b.}$$

Ex. 8. Find the approximate value of $1\cdot05\times1\cdot07$

Here we have

$$(1+0\cdot05)(1+0\cdot07)=1+0\cdot05+0\cdot07=1\cdot12$$

More accurately $(1\cdot05)(1\cdot07)=1+0\cdot05+0\cdot07+0\cdot0035=1\cdot1235.$

Hence, the result obtained by the approximate method is true to the third significant figure.

Similarly, when a and b are small compared with 1,

$$\mathbf{(1+a)^n(1+b)^m=1+na+mb.}$$

We collect the preceding approximation formulae for reference and add others which may be proved in a similar manner.

$$(1 \pm a)^n = 1 \pm na$$

$$(1 \pm a)(1 \pm b) = 1 \pm a \pm b$$

$$(1 \pm a)(1 \pm b)(1 \pm c) \quad = 1 \pm a \pm b \pm c \ldots$$

$$\frac{1}{(1 \pm a)} = 1 \mp a.$$

$$\frac{1}{(1 \pm a)^n} = 1 \mp na.$$

On degree of accuracy.—In the various arithmetical processes of multiplication, division, involution, and evolution, the numbers which are dealt with are usually known to be "correct" to a certain number of significant figures, and it is frequently necessary to ascertain to what number of significant figures a result such as a product or quotient is accurate.

Thus, for example, to find the product of 3·54 and 2·36, it being given that the decimals are correct to the second place It follows that the four decimal places which are obtained in the product are not necessarily correct. Thus, 3·54 means that the number lies between 3·535 and 3·545, and 2·36 lies between 2·355 and 2·365. Hence, the product will lie between 3·535 × 2·355 and 3·545 × 2·365, *i e* between 8·324925 and 8·383925 The product of the given numbers is 3·54 × 2·36 = 8·3544, but in the two extreme cases the result may be expressed as 8·32 or 8·38. Hence the four decimal figures cannot be retained. The result is correct only so far as the whole number is concerned, and at the most we can only retain one decimal place in the result

Hence, in calculating the area of a given figure from two measured lengths, say in inches, it follows that if the measurement be such that an error of 0·01 of an inch is possible, then care is necessary to avoid giving a result which is apparently far more accurate than the given data will supply.

So, too, in dealing with the square, cube, or higher power, of a number, the result must not indicate greater accuracy

than is obtainable from the given data As an example, the area of a circle is given by $\frac{\pi}{4}d^2$, where d is the diameter. If the diameter is 0 08, the area, true to five significant figures, is 0·0050276; but, if d is slightly greater or less than the given amount, the corresponding area is greater or less Thus, if d is 0·079, the area is 0·0049017; and, if 0 081 is 0 005153, and hence, if there is any uncertainty in the second significant figure, not more than one significant figure can be retained in the answer

Assuming d to denote a measured length, and therefore probably slightly in error, it will be absurd to use an accurate value of π This constant has been calculated to over seven hundred significant figures; its value is 3 1416 to five significant figures, and this number is usually sufficiently exact for all practical purposes A good value to use for nearly all practical calculations, indeed, is the number $\frac{22}{7}=3\ 1428$. The number 3 142 is usually used with four-figure logarithms, and it should be noticed that there are comparatively few calculations outside the range of four-figure logarithms.

Ex 9 Let x denote the diameter of a circle. A small error in the measured value of x may be denoted by δx. Calculate the proportional error in the area

For an alteration in the diameter denoted by δx the corresponding change in the area may be denoted by δA

$$A=\frac{\pi}{4}x^2 \quad \ldots \quad \text{(i)}$$

Also $$A+\delta A=\frac{\pi}{4}(x+\delta x)^2=\frac{\pi}{4}\{x^2+2x\delta x+(\delta x)^2\}. \quad \text{(ii)}$$

As δx is a small quantity, its square will be too small to affect the result

Subtracting (i) from (ii) we obtain

$$\delta A=\frac{\pi}{4}(2x\delta x)+\frac{\pi}{4}(\delta x)^2 \quad \text{(iii)}$$

Dividing (iii) by (i) and omitting the last term as being too small to affect the result

the percentage error in the calculated result is twice that made in the measurement of x.

Ex. 10. Let x denote the radius of a circle;

$$\text{the area } y=\pi x^2 \quad \ldots \ldots \ldots \text{(i)}$$

Let the radius increase by an amount δx, then the increase in the area is given by

$$y+\delta y=\pi(x+\delta x)^2=\pi\{x^2+2x\delta x+(\delta x)^2\}, \quad \ldots \text{(ii)}$$

or
$$y+\delta y=\pi x^2+\{2x\delta x+(\delta x)^2\}\pi.$$

Subtract (i) from (ii),

$$\therefore \delta y=\pi\{2x\delta x+(\delta x)^2\};$$

$$\therefore \delta y=2\pi x\delta x+\pi(\delta x)^2$$

Now if δx is exceedingly small, the increase in the area is simply the circumference of a circle of radius x multiplied by the change of radius.

Ex. 11 Let V denote the volume of a sphere of diameter x.

Then
$$V=\frac{\pi}{6}x^3, \quad \ldots \text{(i)}$$

also
$$V+\delta V=\frac{\pi}{6}(x+\delta x)^3=\frac{\pi}{6}\{x^3+3x^2\delta x+3x(\delta x)^2+(\delta x)^3\}$$

As δx is small, we need only retain the first two terms

Subtracting (i) from (ii),

$$\delta V=\frac{\pi}{6}(3x^2\delta x) \quad \ldots \text{(iii)}$$

Dividing (iii) by (i),

$$\frac{\delta V}{V}=\frac{3\delta x}{x}$$

Hence, the percentage error in the calculated volume is three times that made in the measurement of the diameter. In each of the preceding cases, certain terms have been rejected when such terms were small in comparison with a larger one. It will be found that, if an exact determination of the numerical value of such terms is made, no appreciable effect is produced in the result. It is important that this should be verified by the student It clearly applies in the preceding cases, and it may be shown to apply always when, as in raising a number to a given power or extracting a root, one or two terms of a series are sufficient.

Ex. 12. Find the first five terms of the square root of $1+x$, and use the result to obtain the square root of 101.

$$\sqrt{1+x}=(1+x)^{\frac{1}{2}}.$$

Therefore, by using the binomial theorem, we obtain

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4+\quad;$$

$$\therefore\ \sqrt{101}=\sqrt{(100+1)}=10\sqrt{(1+\tfrac{1}{100})}$$

$$=10(1+\tfrac{1}{200}-\tfrac{1}{80000}+\tfrac{1}{16000000}-\text{ etc.})$$

$$=10(1+0{\cdot}005-0{\cdot}0000125+0{\cdot}0000000625)$$

$$=10{\cdot}0049875.$$

By the approximate rule $(1+x)^n=1+nx$,
we obtain $10(1+0{\cdot}005)=10{\cdot}005.$

Zero and infinity.—Probably one of the greatest difficulties met with by a beginner is the meaning to be attached to the words zero and infinity. He is probably familiar with two meanings which may be attached to the former. Thus, in *reference to numbers we might say* $4-4$ *is zero, meaning* that by the subtraction of four from four we obtain a result which has no magnitude. Another form may be roughly shown by considering $4-3{\cdot}999\dot{}\ldots$, in which the difference between the two magnitudes may be made exceedingly small; or, as it is often expressed, when the magnitude of the number representing the difference is made indefinitely small such a quantity may be called zero. In a similar manner, if x and x' are two points on a curve and close together, the distance apart may be indicated either by $x'-x$ or by δx, where δx denotes a small increment of x, which may be either positive or negative. Again, if one number be multiplied by another, the product becomes less and less as one of the numbers diminishes; hence, $b\times 0=0$, or 0 is the limit of bx when x becomes 0.

It is important, also, to understand clearly what is meant by "infinity." Thus, 1 divided by $\frac{1}{100}$ is 100. Similarly, 1 divided by $\frac{1}{1000000}$ is one million. By diminishing the denominator, the result may be made of any magnitude. Hence, as the divisor approaches 0, the quotient becomes an exceedingly great number, and when the denominator is actually 0, the quotient is said to have an infinitely large value, or to be infinite (written as ∞).

The tangent of an angle is the ratio of the sine to the cosine of the angle, or $\tan\theta = \frac{BC}{AB}$ (p. 15); when the angle approaches 90°, the base AB (Fig. 106) becomes exceedingly small; the height becomes equal to the radius of the circle when the angle is 90° and the base is 0; $\therefore \tan 90° = \infty$. Similarly, $\operatorname{cosec}\theta = \frac{AC}{BC}$; as the angle θ approaches 0°, the side BC becomes indefinitely small, and in the limit, when the angle becomes 0°, the side BC is zero, and $\operatorname{cosec} 0° = \infty$ Conversely, as the value of a fraction diminishes by increasing the denominator, it follows that when the denominator becomes indefinitely great, the value of the fraction or its limit is 0. Thus, the value of the fraction $\frac{a}{x}$, when x becomes indefinitely great, is zero

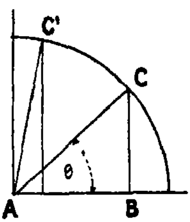

FIG 106

Undetermined forms.—When given values are substituted for x in a fraction, the expression sometimes assumes the form $\frac{0}{0}$, known as an **undetermined form.** There are various methods which may be used to ascertain the value of such an expression. One consists in writing the given expression in factors, removing factors common to both numerator and denominator, and in this manner the factor which reduces the numerator and denominator to the undetermined form may be eliminated.

Ex. 13. Find the value when $x=2$ of the fraction

$$\frac{2x^3 - 7x^2 + 12}{x^3 - 7x + 6}.$$

Substituting the value $x=2$, the given fraction assumes the form $\frac{0}{0}$. Writing the given expression in the form of factors, we have

$$\frac{2x^3 - 7x^2 + 12}{x^3 - 7x + 12} = \frac{(x-2)(2x^2 - 3x - 6)}{(x-2)(x^2 + 2x - 3)}.$$

Cancel the common factor $x - 2$, then $\frac{2x^2 - 3x - 6}{x^2 + 2x - 3}$, and this, when

$x=2$, becomes $\qquad = -\frac{4}{5}$

Limits.—The undetermined form $\frac{0}{0}$ may be used to illustrate the meaning of a *limit*. *Thus, to find the limit of* $\frac{a^2-b^2}{a-b}$, when b approaches to the value of a and ultimately becomes equal to it

So long as b differs from a, the given expression has a definite value. When b becomes equal to a, the expression assumes the form $\frac{0}{0}$. But as $a^2-b^2=(a+b)(a-b)$, we obtain

$$\frac{a^2-b^2}{a-b}=a+b \text{ by division}$$

When $b=a$, this becomes $2a$.

It is important to bear in mind that $\frac{0}{0}$ may have any value whatever.

Ex. 14. Let $y=x^2$, ... (i)
and let x receive a slight increment denoted by δx, then y becomes $y+\delta y$; $y+\delta y=(x+\delta x)^2=x^2+2x\delta x+(\delta x)^2$. (ii)

Subtract (i) from (ii) and divide by δx;

$$\frac{\delta y}{\delta x}=2x+\delta x \qquad \text{(iii)}$$

When the numerical values of x and δx are known, the value of $\frac{\delta y}{\delta x}$ can at once be obtained from (iii) As δx is made smaller and smaller, the value approaches $2x$ in the limit When δx is zero, we obtain $\frac{\delta y}{\delta x}=2x$, from which when x is known the numerical value can be found, also when δx is zero, the preceding is written in the form $\frac{dy}{dx}=2x$.

EXERCISES. XXXIII.

Find the value of

1. $\frac{x^3+x^2-5x+3}{x^4-2x^3-x^2+4x-2}$ when $x=1$.

2. $\frac{(x^2-a^2)^{\frac{3}{2}}+x-a}{(1+x-a)^3-1}$ when $x=a$.

3. $\frac{x^3+2x^2-14x-3}{x^2-x-6}$ when $x=3$.

4. Show that the limit of $\dfrac{a^3-b^3}{a-b}$ when $b=a$ is $3a^2$.

5. Write down and simplify the middle term of the expansion of $\left(1+\dfrac{8x}{15}\right)^6$.

6. Find the third term, also the two middle terms, of $(a+b)^{11}$.

7. Expand $(x \pm a)^6$. 8 $(5-4x)^4$.

9. What is the fifth term of $(x+a)^{16}$?

10. Find by means of a series an approximate value of $\sqrt{7}$

11. Expand $(\sqrt{a} \pm \sqrt{x})^4$.

Numerical value of e.—The value of $\left(1+\dfrac{1}{n}\right)^n$ when n increases without limit is denoted by the letter e, where e is the base of the Naperian logarithms On p 289 we have found that when n is a large number, or in other words when $\dfrac{1}{n}$ is a small number and a is not large compared with n, then $\left(1+\dfrac{1}{n}\right)^a = 1+\dfrac{a}{n}$ approximately

Ex 1 If $n=1000$ and $a=5$,

$$\left(1+\frac{1}{1000}\right)^5 = 1+\frac{5}{1000} = 1{\cdot}005$$

with an error of 1 in 100,000.

The value of $\left(1+\dfrac{1}{1000}\right)^{1000}$ may be obtained by the Binomial Theorem, p. 278, as follows:

$$\left(1+\frac{1}{1000}\right)^{1000} = 1+1000\frac{1}{1000}+1000\frac{(1000-1)}{2}\left(\frac{1}{1000}\right)^2$$

$$+\frac{1000(1000-1)(1000-2)}{2\,.\,3}\left(\frac{1}{1000}\right)^3$$

$$+\frac{1000(1000-1)(1000-2)(1000-3)}{2\,.\,3\ 4}\left(\frac{1}{1000}\right)^4$$

$$+\text{etc}$$

$$=1+1+\frac{1}{2}\left(1-\frac{1}{1000}\right)+\frac{1}{6}\left(1-\frac{3}{1000}\right)+\frac{1}{24}\left(1-\frac{6}{1000}\right)$$

$$+\text{etc.},$$

neglecting such terms as $\dfrac{2}{6}\left(\dfrac{1}{1000}\right)^2$, $\dfrac{11}{24}\left(\dfrac{1}{1000}\right)^3$, etc.

Hence, $$\left(1+\frac{1}{1000}\right)^{1000}=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720},$$

neglecting terms $\frac{1}{2000}$, $\frac{3}{6000}$, $\frac{6}{24000}$, etc.;

$$\therefore \left(1+\frac{1}{1000}\right)^{1000}=2\cdot718$$

with an error of about 1 in 2000.

From the preceding it will be seen that, if a approaches equality with n, the equation $\left(1+\frac{1}{n}\right)^a=1+na$ is very far from being true.

If the value of n be assumed to be 10000 instead of 1000, then the error in the above expansion of $\left(1+\frac{1}{n}\right)^n$ is found to be less than $\frac{1}{20000}$, and when n is 100000 an error of 1 in 200000, in fact the greater n is made the nearer does $\left(1+\frac{1}{n}\right)^n$ approach the value,

$$1+1+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+\dots\frac{1}{1\,.\,2\,.\,3\,.\,4\dots(r-1)r}+\text{etc.}$$

There is equality when n is made indefinitely great, $\left(1+\frac{1}{n}\right)^n$ is then represented by e. A more formal proof than the preceding may be obtained as follows:

$$\left(1+\frac{1}{n}\right)^n=1+n\times\frac{1}{n}+\frac{n(n-1)}{1\,.\,2}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\frac{1}{n^3}$$
$$+\dots+\frac{n(n-1)(n-2)\dots(n-r+1)}{1\,.\,2\,.\,3\,.\,4\dots(r-1)r}\frac{1}{n^r}+\dots$$
$$=1+1+\frac{1}{1\,.\,2}\left(1-\frac{1}{n}\right)+\frac{1}{1\,.\,2\,.\,3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\text{ etc.};$$

and when n is indefinitely great this becomes

$$1+1+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\dots\frac{1}{1\,.\,2\,.\,3\dots r}+\text{ etc.};$$

this is denoted by e, where $e=2\cdot718282\dots$.

This series will give the numerical value of e to any degree of accuracy required.

Ex. 2. Calculate the numerical value of e to five decimal places.

$$1+1+\frac{1}{2}=2{\cdot}500000,\quad \frac{1}{2\,.\,3}=0{\cdot}166666,\quad \frac{1}{2\,.\,3\,.\,4}=0{\cdot}041666,$$

$$\frac{1}{2\,.\,3\,.\,4\,.\,5}=0{\cdot}008333,\quad \frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6}=0{\cdot}001388,$$

$$\frac{1}{\underline{|7}}=0{\cdot}000198,\quad \frac{1}{\underline{|8}}=0{\cdot}000024,\quad \frac{1}{\underline{|9}}=0{\cdot}000003,$$

by addition the numerical value of e is 2·718282.

The symbol $\underline{|7}$, which is read as *factorial seven*, is a convenient term to denote the product of $1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7$. Similarly $\underline{|8}$ denotes $1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8$, etc., $\underline{|r}$ denotes $1\,.\,2\,.\,3\,.\,4 \ldots (r-1)r$.

Expansion of powers of e.—We may now proceed to obtain a series which will enable the values of any power of e such as e^x to be obtained.

Since $e=\left(1+\frac{1}{n}\right)^n$ when n is indefinitely great, e^x is the value of $\left(1+\frac{1}{n}\right)^{nx}$ when n is indefinitely great.

By the Binomial Theorem

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{nx} &= 1+nx\frac{1}{n}+\frac{nx(nx-1)}{\underline{|2}}\frac{1}{n^2}+\ldots \\ &\quad +\frac{nx(nx-1)(nx-2)}{\underline{|3}}\frac{1}{n^3}+\ldots \\ &\quad +\frac{nx(nx-1)(nx-2)(nx-3)\ldots(nx-r+1)}{\underline{|r}}\frac{1}{n^r} \\ &\quad +\ldots\ldots\ldots\ldots \\ &= 1+x+\frac{x^2}{\underline{|2}}\left(1-\frac{1}{nx}\right)+\frac{x^3}{\underline{|3}}\left(1-\frac{1}{nx}\right)\left(1-\frac{2}{nx}\right) \\ &\quad +\ldots\ldots\ldots\ldots \\ &\quad +\frac{x^r}{\underline{|r}}\left(1-\frac{1}{nx}\right)\left(1-\frac{2}{nx}\right)\left(1-\frac{3}{nx}\right)\ldots\left(1-\frac{r-1}{nx}\right) \\ &\quad +\ldots\ldots\ldots\ldots\end{aligned}$$

When n is increased indefinitely

$$\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\quad+\frac{x^r}{\underline{|r}}+\ldots,$$

there being an infinite number of terms,

$$\text{or}\qquad e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots+\frac{x^r}{\underline{|r}}+\ldots$$

Hence it follows at once that

$$e^{ax}=1+ax+\frac{a^2x^2}{\underline{|2}}+\frac{a^3x^3}{\underline{|3}}+\ldots+\frac{a^rx^r}{\underline{|r}}+\ldots\ldots\ldots\ldots\text{(i)}$$

Ex. 3. Calculate to four decimal places the value of e^x when $x=1{\cdot}2$.

From (i) we obtain, where $a=1$,

$$1+x=2{\cdot}20000,\quad \frac{x^2}{2}=0{\cdot}72000,\quad \frac{x^3}{2\,3}=0{\cdot}28800,$$

$$\frac{x^4}{\underline{|4}}=0{\cdot}08640,\quad \frac{x^5}{\underline{|5}}=0{\cdot}02064,\quad \frac{x^6}{\underline{|6}}=0{\cdot}00413,$$

$$\frac{x^7}{\underline{|7}}=0{\cdot}00071,\quad \frac{x^8}{\underline{|8}}=0{\cdot}00011,\quad \text{the sum is } 3{\cdot}32000.$$

Other values of x, *e g.* 0·4, 0·8, 1·6, 2·0, etc., may in like manner be assumed and the corresponding values of e^x obtained.

From the series for e^x it will be obvious that when x is 0, $e^0=1$.

When x is indefinitely great, or (as usually expressed) when x is infinite, e^x is infinite;

$$\therefore\quad e^\infty=\infty.$$

Also, since $e^{-x}=\frac{1}{e^x}=0$, when x is infinite, e^x has a range of positive values from zero to ∞, as x changes from $-\infty$ to $+\infty$. That its value cannot be negative if x is real may be seen from the graph on p. 141.

Expansion a^x.—The series for a^x is readily deduced from that of e^x.

Since e^c can have any positive value from zero to infinity, it follows that if a is any real positive quantity whatever, we can always find c, so that $e^c=a$.

Thus if $a=2$, $c=0{\cdot}693147$ to 6 places of decimals, and $e^{0{\cdot}693147}=2$. In fact, we see from the definition of logarithms, p. 49, $c=\log_e a$, if $e^c=a$.

But $$a^x=e^{cx}=1+cx+\frac{c^2x^2}{\lfloor 2}+\frac{c^3x^3}{\lfloor 3}+\ldots+\frac{c^rx^r}{\lfloor r}+\ldots;$$

$$\therefore\ a^x=1+x\log_e a+\frac{(x\log_e a)^2}{\lfloor 2}+\ldots+\frac{(x\log_e a)^r}{\lfloor r}+\ldots,$$

and $$a^{bx}=1+bx\log_e a+\frac{(bx\log_e a)^2}{\lfloor 2}+\ldots+\frac{(bx\log_e a)^r}{\lfloor r}+\ldots$$

We collect here for reference the four expansions already obtained.

(*a*) $$\mathbf{e}=\mathbf{1}+\mathbf{1}+\frac{\mathbf{1}}{\lfloor \mathbf{2}}+\frac{\mathbf{1}}{\lfloor \mathbf{3}}+\ldots+\frac{\mathbf{1}}{\lfloor \mathbf{r}}+\ldots;$$

(*b*) $$\mathbf{e}^{x}=\mathbf{1}+\mathbf{x}+\frac{\mathbf{x}^2}{\lfloor \mathbf{2}}+\frac{\mathbf{x}^3}{\lfloor \mathbf{3}}+\ldots+\frac{\mathbf{x}^r}{\lfloor \mathbf{r}}+\ldots;$$

(*c*) $$\mathbf{e}^{ax}=\mathbf{1}+\mathbf{ax}+\frac{\mathbf{a}^2\mathbf{x}^2}{\lfloor \mathbf{2}}+\frac{\mathbf{a}^3\mathbf{x}^3}{\lfloor \mathbf{3}}+\ldots+\frac{\mathbf{a}^r\mathbf{x}^r}{\lfloor \mathbf{r}};$$

(*d*) $$\mathbf{a}^{bx}=\mathbf{1}+\mathbf{bx}\log_e\mathbf{a}+\frac{(\mathbf{bx}\log_e\mathbf{a})^2}{\lfloor \mathbf{2}}+\frac{(\mathbf{bx}\log_e\mathbf{a})^3}{\lfloor \mathbf{3}}+\ldots.$$

The last is the most general of the preceding series; from this one the remaining series may be obtained by giving particular values to b and x, and substituting e for a.

Expansion of $\log_e(1+x)$.—Take the series

$$a^z=1+z\log_e a+\frac{z^2(\log_e a)^2}{\lfloor 2}+\frac{z^3(\log_e a)^3}{\lfloor 3}+\ldots,$$

and let $$a=1+x;$$

$$\therefore\ (1+x)^z=1+z\log_e(1+x)+\frac{z^2}{\lfloor 2}\{\log_e(1+x)\}^2+\ldots.$$

But, by the Binomial Theorem, p. 278,

$$(1+x)^z=1+zx+\frac{z(z-1)x^2}{\lfloor 2}+\frac{z(z-1)(z-2)}{\lfloor 3}x^3+\ldots$$
$$+\frac{z(z-1)(z-2)(z-3)\ \ (z-r+1)}{\lfloor r}x^r+.$$
$$=1+z\left\{x-\frac{x^2}{2}+\frac{1.2}{\lfloor 3}x^3+\ldots+(-1)^{r-1}\frac{1.2.3\ \ (r-1)}{\lfloor r}x^r+\ldots\right\}$$
$$+z^2\left(\frac{x^2}{2}+\quad\right)+\quad.$$

This is only true when x is less than 1, for the Binomial Theorem is only applicable in such a case. Hence, if x is < 1,

$$1+z\left\{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots\right\}+z^2\left\{\left(\frac{x}{2}\right)^2+\dots\right\}+\dots$$

$$=1+z\log_e(1+x)+\frac{z^2}{\underline{|2}}\{\log_e(1+x)\}^2+\dots$$

for all values of z. Therefore, the coefficient of any power of z on one side of the identity is equal to that of the similar power on the other, provided x is not > 1.

Selecting the coefficients of the first power of z, we obtain the series

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\quad .$$

This holds when x is not greater than unity. But when x is greater than unity it is obviously infinite in value for an infinite number of terms. But $\log_e(1+x)$ is finite, if x is finite; hence the above cannot be true if $x > 1$.

Calculation of logarithms.—From the preceding series it is possible to calculate a table of logarithms to the base e.

Ex. 4. In the preceding series (i) put $x=\frac{1}{2}$, and we obtain

$$\log_e\tfrac{3}{2}=\tfrac{1}{2}-\tfrac{1}{2}(\tfrac{1}{2})^2+\tfrac{1}{3}(\tfrac{1}{2})^3-\tfrac{1}{4}(\tfrac{1}{2})^4+\dots,$$

or $$\log_e 3-\log_e 2=\tfrac{1}{2}-\tfrac{1}{2}(\tfrac{1}{2})^2+\tfrac{1}{3}(\tfrac{1}{2})^3-\tfrac{1}{4}(\tfrac{1}{2})^4+$$

$$=0{\cdot}549305-0{\cdot}143840;$$

$$\therefore \log_e 3-\log 2=0{\cdot}405465.$$

In a similar manner, the series may be used to calculate the numerical values of $\log_e 2$, $\log_e 3$, .. .

Thus, substituting $x=\frac{1}{3}$ in the series for $\log(1+x)$, we obtain

$$\log_e 4-\log_e 3=0{\cdot}287682,$$

$$\therefore 2\log_e 2-\log_e 3=0{\cdot}287682;$$

also $$\log_e 3-\log_e 2=0{\cdot}405465,$$

$$\therefore \text{by addition } \log_e 2=0{\cdot}693147,$$

and $$\log_e 3=1{\cdot}098612;\ \text{also } \log_e 4=1{\cdot}386294.$$

Other selected values may be calculated in like manner.

The preceding method is much too laborious for general use in calculations. More convenient formulae may be obtained as follows:

Thus, $$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots;$$

$$\therefore \log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots.$$

The latter is obtained from the former by writing $-x$ for x

Subtracting,

$$\therefore \log_e(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right),$$

or $$\log_e\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right). \quad . \quad . \quad \text{(ii)}$$

If x is small it is only necessary to retain and calculate the values of two or three terms in the series (ii).

Ex. 5. Given $\log_e 9 = 2\cdot197224$, find the value of $\log_e 11$.

If $x = \frac{1}{10}$, $$\log_e\frac{1+x}{1-x} = \log_e 11 - \log_e 9$$
$$= 2\left\{\frac{1}{10} + \frac{1}{3 \times 10^3} + \dots\right\}$$

A series in which it is necessary to retain only a few terms

It will be obvious that if a series for $\log_e(n+1) - \log_e n$ can be obtained in which the successive terms in the series decrease very rapidly, then it will be possible, when $\log_e n$ is known, to obtain $\log_e(n+1)$, and therefore the logarithms of all numbers consecutively.

Now $$\log_e(n+1) - \log_e n = \log_e\frac{1+n}{n}.$$

Let $$\frac{1+n}{n} = \frac{1+x}{1-x},$$

$$\therefore (1+n)(1-x) = n(1+x); \quad \therefore x = \frac{1}{2n+1}.$$

Now substitute this value of x obtained in the series for $\log_e\frac{1+x}{1-x}$;

$$\log_e\frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\}$$

A series in which the successive terms decrease very rapidly.

Calculation of common logarithms.—To calculate common logarithms or logarithms to base 10, we may, as indicated on p. 54, divide the logarithm of a number to base e by $\log_e 10$.

Thus $\log_e 2 = 0{\cdot}69315$ and $\log_e 10 = 2{\cdot}30258$;

$$\therefore \log_{10} 2 = \frac{0{\cdot}69315}{2{\cdot}30258} = 0{\cdot}30103.$$

Proceeding in this manner it would be possible to change the logarithms of all numbers calculated to base e into common logarithms.

The number $\dfrac{1}{\log_e 10} = 0{\cdot}4342945$ is called the **modulus** of the common system of logarithms, and is often represented by the letter μ.

Thus, the series for $\log_e \dfrac{n+1}{n}$ and the value of μ enables us to calculate common logarithms directly, for

$$\log_{10} \frac{n+1}{n} = \mu \log_e \frac{n+1}{n}.$$

Hence, $$\log_{10} \frac{n+1}{n} = 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \dots \right\}.$$

The work is further simplified by the fact that

$$\log_{10}(10^r \times n) = r + \log_{10} n.$$

Thus $$\log_{10} 1{\cdot}0561 = \log_{10} 10561 - 4,$$

since $$1{\cdot}0561 = \frac{10561}{10^4}.$$

Ex. 6. Calculate $\log_{10} 10{\cdot}001$ to 7 decimal places.

$$\log_{10} 10{\cdot}001 = \log 10{,}000 + \frac{2 \times 0{\cdot}4342945}{20{,}001}$$
$$= 4 + 0{\cdot}0000434;$$
$$\therefore \log 10{\cdot}001 = 1{\cdot}0000434.$$

Similarly, $$\log_{10} 10{,}002 = \log 10{,}001 + \frac{2 \times 0{\cdot}4342945}{20{,}003}$$
$$= 4{\cdot}0000868.$$

EXERCISES. XXXIV.

1. The values of $\log_e 2$, $\log_e 3$, are $0{\cdot}69315$, and $1{\cdot}09861$ respectively, calculate and tabulate the logarithms of 5, 6, 7, 8, 9, and 10 to base e in each case to 5 significant figures.

2. Given $\log_e 30 = 3{\cdot}401197$, calculate to 5 significant figures the numerical values of the logarithms of 31, 32, 33, 34, 35, 36, 37, 38, 39, and 40.

3. Find series for the expressions

$$\frac{e^x + e^{-x}}{2}, \quad \frac{e^x - e^{-x}}{2}, \quad \frac{e^{ax} + e^{-ax}}{2}, \quad \frac{e^{ax} - e^{-ax}}{2}.$$

4. Taking $\log_e 1{\cdot}1 = 0{\cdot}09531$, test the identity

$$(1{\cdot}1)^2 = 1 + 2\log_e 1{\cdot}1 + \frac{(2\log_e 1{\cdot}1)^2}{\lfloor 2} + \quad + \frac{(2\log_e 1{\cdot}1)^r}{\lfloor r}$$

to four decimal places

5. Given $\log_{10} 4{\cdot}1110 = 0{\cdot}6139475$, calculate the logarithms of numbers of 6 significant figures between 4·1110 and 4·1120.

6. Take $\log_{10} 3420$ from the tables at the end of the book and calculate logs of numbers between 3420 and 3430 to at least 4 decimal places. Compare your answers with the tables.

Hint. Use $n = 3420$, not 34,200 as before;

$$\therefore \log_{10} 3421 = \log 3420 + \frac{2\mu}{2(3420) + 1}.$$

CHAPTER XIV.

RATE OF INCREASE. SIMPLE DIFFERENTIATION.

Rate of increase.—Most students are probably familiar with what is meant by such a statement as the following.—The population of a country in 1901 was 3,000,000 in excess of that in 1891, thus giving an **average rate of increase** of 300,000 per year during the ten years. The calculation involved is simply the increase of population for the 10 years divided by 10, and this gives what is called the average rate of increase per year This average rate of increase, though useful to the statistician, is not sufficiently definite for mathematical purposes. Such a rate does not, for instance, give the rate of increase for any one year, this might be 200,000 during 1898 and 400,000 during 1899 without altering the average rate during the ten years.

Probably a better illustration is obtained from a table such as the following, in which the relation between y and x is $y=x^2$, and in which for values of x corresponding values of y are given

From such a table we are able to ascertain the average and also the actual, rate of increase of a given quantity.

x	4000	4·0001	4 001	4·01	4·1
y	16·0000	16 0008001	16·008001	16·0801	16·81

The amount by which one value of x has increased, to form a second value x, is called an **increment** of x. Thus, referring to the table and subtracting 4·0 from 4·1 we obtain 0·1; this is the increment of x which is being considered; and

$16{\cdot}81-16{\cdot}0=0{\cdot}81$ is the corresponding increment of y. The average rate of increase of y, as x increases from 4·0 to 4·1, is the increment of y divided by the corresponding increment of x, and is equal to

$$\frac{0{\cdot}81}{0{\cdot}1}=8{\cdot}1.$$

Taking other values from the table, we have, between $x=4{\cdot}0$ and 4·01, the ratio:

$$\frac{\text{increment of } y}{\text{increment of } x}=\frac{0{\cdot}0801}{0{\cdot}01}=8{\cdot}01.$$

Between $x=4{\cdot}0$ and $x=4{\cdot}001 \quad =\dfrac{0{\cdot}008001}{0{\cdot}001}=8{\cdot}001$

Between $x=4{\cdot}0$ and $x=4{\cdot}0001=\dfrac{0{\cdot}00080001}{0{\cdot}0001}=8{\cdot}0001.$

Thus, the average rate of increase of y is a variable quantity which depends on the magnitude of the increment of x. Further, as the increment is diminished, the corresponding increment of y also diminishes, and the average rate approaches a value 8. The approximation becomes closer and closer as the increment of x is diminished, and ultimately, when the increment of x is made indefinitely small, the ratio has the value 8, and this is the actual rate of increase of y when $x=4$.

The value 8 is then said to be the **limit** of the ratio of the increment of y to the corresponding increment of x

As the expression "increment of y" occurs frequently, the symbol δy is used to denote an increment of y, and the above ratio is written $\dfrac{\delta y}{\delta x}$.

The expression "the limit of $\dfrac{\delta y}{\delta x}$ when δx diminishes without limit" is written in the form

$$\operatorname*{Lt}_{\delta x=0}\frac{\delta y}{\delta x}.$$

The final value of δx *will be zero*, and the result obtained is called the **differential coefficient of y.** This is the definition in its algebraic form of a differential coefficient.

Comparing this, step by step, with the example given, we obtain for one particular case

$$\left.\begin{array}{l}\delta y=0\cdot81\\ \delta x=0\cdot1\end{array}\right\},$$

the ratio $\frac{\delta y}{\delta x}$ having a numerical value of 8·1 or $8+\delta x$.

Again, for a second case, $\delta y=0\cdot801$ and $\delta x=0\cdot01$,

or $$\frac{\delta y}{\delta x}=\frac{0\cdot801}{0\cdot01}=8\cdot01, \text{ or } 8+\delta x,$$

and so on as far as possible.

It is obvious, however, that we may proceed to make δx less and less, and shall not come to a stop until it is absolutely zero. When this occurs, $8+\delta x$ becomes $8+0$ or 8—a perfectly definite result, which does not depend on the increment taken. Or, in other words, we have reached a limit to the value of $\frac{\delta y}{\delta x}$, and we call it a differential coefficient, writing it $\frac{\delta y}{\delta x}$.

It must be carefully noticed that in $\frac{dy}{dx}$, $\frac{d}{dx}$ is a symbol of an operation just as $\div$ indicates division, or $\times$ indicates multiplication, and therefore it *does not mean* $d\times x$ and $d\times y$; the symbol $\frac{dy}{dx}$ simply indicates a rate of increase.

The relation between two variables x and y from which the preceding numbers may be calculated being given by

$$y=x^2. \qquad \text{(i)}$$

Let $x+\delta x$ denote a slightly larger value of x, and $y+\delta y$ the corresponding value of y. Then we obtain from (i), by substitution,

$$y+\delta y=(x+\delta x)^2$$
$$=x^2+2x\delta x+(\delta x)^2. \qquad \text{(ii)}$$

Subtract (i) from (ii),

$$\delta y=2x\delta x+(\delta x)^2.$$

Divide both sides by δx;

$$\therefore \frac{\delta y}{\delta x}=2x+\delta x.$$

Comparison with the preceding tabulated numbers will explain the meaning when $x=4$ of $\frac{\delta y}{\delta x}=8+\delta x$,
and for the reasons already given when δx becomes zero we write $\frac{dy}{dx}$ instead of $\frac{\delta y}{\delta x}$, and say that the differential coefficient of y is 8 when x has the value 4.

Ex. 1. From the definition

$$\frac{dy}{dx}=\text{Lt}_{\delta x=0}\frac{\delta y}{\delta x},$$

find $\frac{dy}{dx}$ when $$y=10+5x+3x^2. \quad \ldots \quad (\text{i})$$

The equation (i) must be true for all values of y and x.

Hence $$y+\delta y=10+5(x+\delta x)+3(x+\delta x)^2$$
$$=10+5x+5\delta x+3x^2+6x\delta x+3(\delta x)^2. \quad \ldots \quad (\text{ii})$$

Subtracting (i) from (ii),

$$\delta y=5\delta x+6x\delta x+3(\delta x)^2,$$

or $$\frac{\delta y}{\delta x}=5+6x+3\delta x.$$

Now make $\delta x=0$; this also makes $\frac{\delta y}{\delta x}$ become $\frac{dy}{dx}$, and we obtain

$$\text{Lt}_{\delta x=0}\frac{\delta y}{\delta x}=\left[\frac{dy}{dx}\right]=5+6x. \quad \ldots \quad (\text{iii})$$

Expressing (iii) in words we may say "The limit of the ratio of the increment of y to the increment of x, when the latter is made zero, is called the differential coefficient of y with respect to x, and is equal, in the case considered, to $5+6x$."

Ex. 2. Show that when

$$y=x^3,\ u=x^4,\ v=5x^2,$$

then $$\frac{dy}{dx}=3x^2,\ \frac{du}{dx}=4x^3,\ \frac{dv}{dx}=10x;$$

also when $y=ax^3$, $$\frac{dy}{dx}=3ax^2.$$

$\frac{dy}{dx}$ has been defined as $\text{Lt}_{\delta x=0}\frac{\delta y}{\delta x}$,

and in order to find its actual value the relation between x and y must be known. This is expressed by saying that y is some function of x, or $$y=f(x) \ldots \ldots \quad (\text{i})$$

As before $y+\delta y$ and $x+\delta x$ are simultaneous values;

$$\therefore\ y+\delta y=f(x+\delta x). \qquad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Subtract (i) from (ii);

$$\therefore\ \delta y=f(x+\delta x)-f(x).$$

Substitute this value in the definition above, and

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{f(x+\delta x)-f(x)}{\delta x}.$$

This is the usual expression for defining a differential coefficient and is more convenient for use.

Ex. 3. Given that $y=3x^3+9x$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{\{3(x+\delta x)^3+9(x+\delta x)\}-(3x^3+9x)}{\delta x}$$

$$=\mathrm{Lt}_{\delta x=0}\frac{3\{3x^2\delta x+3x(\delta x)^2+(\delta x)^3\}+9\delta x}{\delta x}$$

$$=\mathrm{Lt}_{\delta x=0}\{9x^2+9x\delta x+3(\delta x)^2+9\}.$$

Apply the limiting condition, *i.e.* put $\delta x=0$, and $\quad \dfrac{dy}{dx}=9x^2+9.$

The differential coefficients of certain expressions such as $y=x^n$, $y=\sin x$, etc., are of the utmost importance; the results when obtained should be committed to memory.

Differential coefficient x^n.—If $y=x^n$, then, from the definition just given, the average rate of increase of y with respect to x is

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{(x+\delta x)^n-x^n}{\delta x}$$

$$=\mathrm{Lt}_{\delta x=0}\frac{x^n\left(1+\dfrac{\delta x}{x}\right)^n-x^n}{\delta x}$$

$$=\mathrm{Lt}_{\delta x=0}\frac{x^n\left\{\left(1+\dfrac{\delta x}{x}\right)^n-1\right\}}{\delta x}.$$

Since $\dfrac{\delta x}{x}$ is <1 we may apply the Binomial Theorem (p. 278) to the expansion of $\left(1+\dfrac{\delta x}{x}\right)^n$, and therefore

$$\left(1+\frac{\delta x}{x}\right)^n=1+\frac{n\delta x}{x}+\frac{n(n-1)}{\underline{|2}}\left(\frac{\delta x}{x}\right)^2+\frac{n(n-1)(n-2)}{\underline{|3}}\left(\frac{\delta x}{x}\right)^3+\ldots;$$

$$\therefore \left(1+\frac{\delta x}{x}\right)^n - 1 = \frac{n\delta x}{x} + \frac{n(n-1)}{\underline{|2}}\left(\frac{\delta x}{x}\right)^2 + \text{etc},$$

and $$\frac{\left(1+\frac{\delta x}{x}\right)^n - 1}{\delta x} = \frac{n}{x} + \frac{n(n-1)}{\underline{|2}}\frac{1}{x^2}(\delta x) + \text{etc}$$

The remaining terms will contain increasing powers of δx as multipliers, and will therefore disappear in the limit, when δx is made zero

Hence, the value of $\frac{dy}{dx}$ is $x^n \times \frac{n}{x} = nx^{n-1}$;

$$\therefore \text{ when } y = x^n, \quad \frac{dy}{dx} = nx^{n-1}.$$

Differential coefficient of sin x.—To obtain the differential coefficient when $y = \sin x$, we have, by definition,

$$\frac{dy}{dx} = \text{Lt}_{\delta x = 0}\frac{\sin(x+\delta x) - \sin x}{\delta x},$$

and by Trigonometry (p. 28),

$$\sin(x+\delta x) - \sin x = 2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2};$$

$$\therefore \frac{dy}{dx} = \text{Lt}_{\delta x = 0}\frac{\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}. \quad \ldots \text{(i)}$$

Now the value of $\frac{\sin A}{A}$, when A is very small and measured in radians, is very nearly unity, and when A is zero the ratio is exactly 1;

$$\therefore \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} = 1, \text{ also } \cos\left(x+\frac{\delta x}{2}\right) = \cos x, \text{ when } \delta x \to 0$$

Hence, $\frac{dy}{dx} = \cos x$ from (i).

Differential coefficient of cos x.—The value of $\frac{dy}{dx}$, when $y = \cos x$, may be obtained in a similar manner to the preceding;

$$\therefore \frac{dy}{dx} = \text{Lt}_{\delta x = 0}\frac{\cos(x+\delta x) - \cos x}{\delta x},$$

and by Trigonometry (p 28), this

$$=\mathrm{Lt}_{\delta x=0}\frac{-\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\frac{\delta x}{2}};$$

$$\therefore\ \frac{dy}{dx}=-\sin x,$$

since
$$\mathrm{Lt}_{\delta x=0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=1.$$

Differential coefficient of e^x.—The differential coefficient of $y=e^x$ may be obtained as follows.

By definition $\frac{dy}{dx}$ is the limiting value of

$$\frac{dy}{dx}=\frac{e^{x+\delta x}-e^x}{\delta x}$$

when δx is made zero,

$$i.e.\ \frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{e^xe^{\delta x}-e^x}{\delta x}$$

$$=e^x\frac{e^{\delta x}-1}{\delta x}.$$

But, as on p 292,

$$e^{\delta x}=1+\delta x+\frac{(\delta x)^2}{\underline{|2}}+\ldots,$$

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\,e^x\times\frac{\left(1+\delta x+\frac{\delta x^2}{\underline{|2}}+\ldots\right)-1}{\delta x}$$

$$=e^x\left\{1+\frac{\delta x}{\underline{|2}}+\frac{(\delta x)^2}{\underline{|3}}+\ldots\right\}.$$

Now, when δx becomes zero, all terms in the brackets, except the first, disappear;

$$\therefore\ \frac{dy}{dx}=e^x.$$

The last result may be obtained as follows:
Let $y=e^x$.

Now, $$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots \text{ (see p. 292)};$$

$$\frac{dy}{dx}=\frac{d}{dx}(e^x)=\frac{d}{dx}\left(1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots\right).$$

Differentiating,

$$\therefore \frac{d}{dx}(e^x)=0+1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots.$$

It will be noticed that the series obtained by differentiation is identical with the original series;

$$\therefore \frac{d}{dx}(e^x)=e^x.$$

In other words, **the rate of increase, or differential coefficient, of e^x, is the function itself.** This remarkable result, as indicated on p. 474, is known as the **compound interest law.**

Differentiation of $\log_e x$.—Let $y=\log_e x$;

$$\therefore \frac{dy}{dx}=\text{Lt}_{\delta x=0}\frac{\log_e(x+\delta x)-\log_e x}{\delta x}.$$

But the difference of two logarithms is the logarithm of their quotient (p. 51);

$$\therefore \frac{dy}{dx}=\text{Lt}_{\delta x=0}\frac{\log_e\dfrac{x+\delta x}{x}}{\delta x}$$

$$=\text{Lt}_{\delta x=0}\frac{\log_e\left(1+\dfrac{\delta x}{x}\right)}{\delta x}.$$

Now, using the expansion for $\log_e\left(1+\frac{\delta x}{x}\right)$ (p. 293), we obtain

$$\frac{dy}{dx}=\text{Lt}_{\delta x=0}\left\{\frac{\delta x}{x}-\frac{1}{2}\left(\frac{\delta x}{x}\right)^2+\frac{1}{3}\left(\frac{\delta x}{x}\right)^3-\dots\right\}\div\delta x$$

$$=\text{Lt}_{\delta x=0}\left\{\frac{1}{x}-\frac{1}{2}\frac{\delta x}{x^2}+\frac{1}{3}\frac{(\delta x)^2}{x^3}-\text{ etc.}\right\}$$

$$=\frac{1}{x}.$$

Hence, the differential coefficient of $\log_e x$ is $\frac{1}{x}$.

Geometrical meaning of $\frac{dy}{dx}$.—In order to make the meaning of $\frac{dy}{dx}$, or of a *rate of increase*, clear, it may be necessary to consider the properties of the tangent line at a given point on a curve, particularly with regard to the angle made by the line with the axis of x, or as it is called the **slope of the line**

If we take a line PQR (Fig 107), its inclination to the axis of x, or the slope of the line, may be measured by several different methods.

A length PR may be measured along the incline and the height of R, RT, above P obtained. Then the ratio $\frac{RT}{PR}$ or $\sin\theta$ is called by surveyors and others, the gradient or the slope of the road. It is usually expressed as a fraction having unity for its numerator, such as $\frac{1}{10}$, $\frac{1}{100}$, etc.

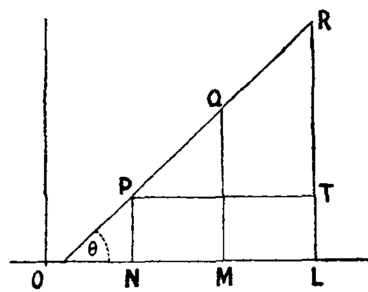

FIG 107.

A much more convenient method for mathematical purposes is given by the ratio of RT to PT;

$$\tan\theta = \frac{RT}{PT}$$

This will, in all the following cases, be called the slope of the line.

Tangent to a curve.—The tangent to a curve at a given point is defined as the straight line touching the curve at the point In the case of a curve which passes through a series of plotted points, the line joining two points on the curve close to each other can be determined by diminishing the distance between them In this manner the approximation to the tangent at a point may be made to any degree of accuracy and the tangent is the limit; *i.e.* when the points forming two consecutive points coincide on the curve.

Slope of a curve.—**The slope of a curve at a given point may be defined as the tangent of the angle (made by the tangent to the curve at that point) with the axis of x.**

Meaning of differential coefficient at a point on a curve.—Suppose PSQ to be a portion of a curve found by plotting $y=f(x)$. Taking the algebraic form of expression for $\frac{dy}{dx}$ and applying it to the geometrical case illustrated in Fig. 108.

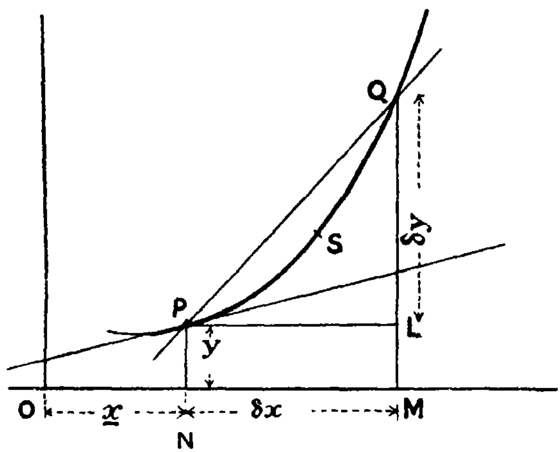

FIG 108

If $y=f(x)$,

then
$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{f(x+\delta x)-f(x)}{\delta x},$$

and since $f(x)=y$ and $f(x+\delta x)=y+\delta y$ it may be written

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{(y+\delta y)-y}{\delta x}$$

Let y denote PN and $QM=y+\delta y$, then NM will be denoted by δx
$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{QM-PN}{NM}$$

But $QM-PN$ is equal to QL and $NM=PL$, whilst

$$\frac{QL}{PL}=\tan\phi.$$

But $\tan\phi$ has been defined as the slope of the line PQ;

$\therefore$ replacing $\frac{QM-PN}{NM}$ by the words "the slope of the line PQ," we obtain

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}, \text{ "the slope of the line } PQ."$$

Now, as δx decreases, *i.e.* as Q approaches nearer and nearer to P, PQ also approximates closer and closer to the tangent PT, and will become the tangent at P when $\delta x=0$, *i.e.*

"$\mathrm{Lt}_{\delta x=0}$, the slope of the line PQ," now becomes the slope of the tangent at P.

Also, as $y=PN$, it follows that the differential coefficient of PN, with respect to x, is equal to the slope of the tangent at P.

Ex. 4. $y=\frac{1}{2}x^2$.

By the algebraic method,

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{\frac{1}{2}(x+\delta x)^2-\frac{1}{2}x^2}{\delta x}$$

$$=x.$$

Now plot the curve from $y=0$ to $y=1$.

This is shown by the curve in Fig. 109, p. 308.

Put the set square in the position indicated in Fig 109, and draw the tangent at the point P as carefully as possible, P being the point for which $x=1$.

Measure the angle θ, and obtain its tangent from Table VI., or measure $\tan\theta$ directly from the figure by making NT equal to unity, and measuring on the vertical scale the length of MT, this is seen to be unity;

$$\therefore\ \tan\theta=\frac{MT}{NT}=\frac{1}{1}=1.$$

We have already found that $\frac{dy}{dx}=x$, and therefore for the point P, where $x=1$, $\frac{dy}{dx}=1$.

In a similar manner, other points on the curve may be selected, and the numerical values of $\frac{dy}{dx}$ can be calculated by

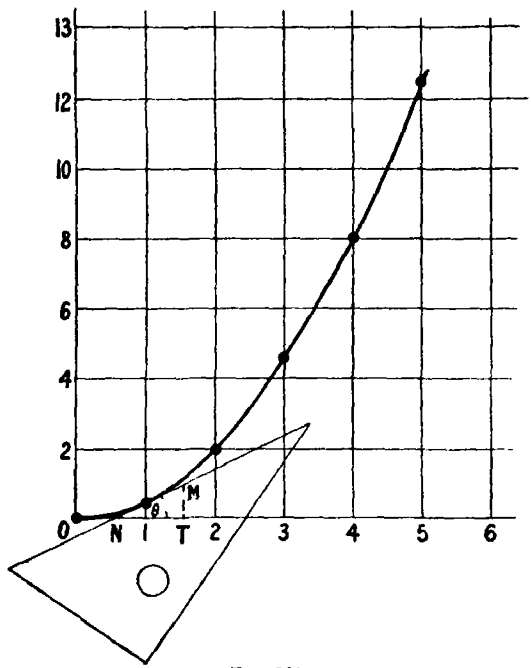

FIG. 109.

measuring the tangent of the angle made between the tangent to the curve and the axis of x.

EXERCISES. XXXV

In each of the following, from the given value of y, find the value of $\frac{dy}{dx}$.

1. $y=x^4+3x^3-x^2+5$
2. $y=Ax^n$.
3. $y=\sin ax$.
4. $y=A\sin ax$.
5. $y=A\cos ax$
6. $y=\sqrt{x^3}$.
7. Find $\frac{ds}{dt}$ from $s=v_0t+\frac{1}{2}at^2$.

8. Illustrate that if $y = \sin x$, then $\frac{dy}{dx} = \cos x$ by working out the following table:

Angle in degrees	Angle in radians	y or $\sin x$	δx	δy	$\frac{\delta y}{\delta x}$	Average value of $\frac{\delta y}{\delta x}$
40°	0·698131	0·642788				
40°·1	0·699877	0·644124				
40°·2	0·701622	0·645458				

9. If $u = \sin x$, and $v = \cos x$, determine by first principles the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$.

Hence, or otherwise, find the value of $\frac{dy}{dx}$ where $y = \tan x$.

10. Determine the values of

$$\frac{d(a \sin bx)}{dx}, \quad \frac{d(a \cos bx)}{dx}, \quad \frac{d(ax^n)}{dx}.$$

11. $u = a\cos(bx + c)$, $v = \log(a + bx)$. Determine the values of

$$\frac{du}{dx}, \frac{dv}{dx}.$$

Find the differential coefficient in each of the following cases:

12. $y = \sqrt{a^2 - x^2}$.

13. $y = \cot x$

14. $y = \log ax$.

15. $y = a^x$.

16. $y = \sin ax^n$

17. $v = \frac{a - t}{t}$.

18. $v = \sqrt{a^2 - t^2}$

19. $y = \log x^2$.

20. $y = 4x^2 + 13x + 4$.

21. $y = 5x^2 - 9x + 2$.

22. $y = x^5 + 4x^3$.

23. $y = 2x^{-\frac{3}{2}}$

24. $pv^{1\cdot 408} = c$, find $\frac{dp}{dv}$.

25. $s = \frac{1}{2}ft^2$, find $\frac{ds}{dt}$

26. $v = ft$, find $\frac{dv}{dt}$.

CHAPTER XV.

DIFFERENTIATION

THE definitions and principles of the preceding chapter are probably sufficient to enable the student to find the rate of increase, or the differential coefficient, of any function with respect to its variable, provided there is sufficient data given with regard to the function.

The labour thus involved may be reduced by the use of certain rules.

[Such rules have an undoubted advantage from a labour-saving point of view; but, as they may in some cases hide the steps in the work, and as it is so easy a matter for a student to use such rules without understanding them, it may be desirable to explain somewhat fully how some of these rules may be obtained.]

Differential coefficient of a constant.—As a constant is, from definition, an invariable quantity, and admits of no variation, it follows that if $y=c$, then δy, which denotes an increase in the value of y, is zero, and, therefore, all values of $\frac{\delta y}{\delta x}$ are zero, and consequently the limit $\frac{dy}{dx}=0$. Now, it will be obvious that $y=c$ denotes a line parallel to the axis of x and at a distance c from it. Hence, the tangent of the inclination, *i.e.* $\frac{dy}{dx}$ is zero.

Differentiation of a sum of functions.—This problem has been illustrated in a former chapter, but the general proof may with advantage be given here.

Let $$y=u+v+w,$$
where u, v, and w are each functions of x; and let $u+\delta u$, $v+\delta v$,

and $w+\delta w$ be the values of these functions when x has become $x+\delta x$.

Then, by definition,

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left\{\frac{(u+\delta u+v+\delta v+w+\delta w)-(u+v+w)}{\delta x}\right\}$$

$$=\mathrm{Lt}_{\delta x=0}\left\{\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x}+\frac{\delta w}{\delta x}\right\}.$$

But, making δx zero, which is an independent operation for each fraction, we obtain

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{\delta u}{\delta x}+\mathrm{Lt}_{\delta x=0}\frac{\delta v}{\delta x}+\mathrm{Lt}_{\delta x=0}\frac{\delta w}{\delta x}.$$

But $\mathrm{Lt}_{\delta x=0}\dfrac{\delta u}{dx}=\dfrac{du}{dx}$, and so on for the others;

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}.$$

This form is most convenient for use, but it is often necessary to use more cumbrous expressions than u, v, and w for functions of the independent variable; and for this reason, the same operations are repeated exactly as follows:

Let $$y=F(x)+f(x)+\phi(x),$$

where $F(x)$, $f(x)$, and $\phi(x)$ denote functions of the variable x and do not contain the variable y; when x becomes $x+\delta x$, then y becomes

$$y+\delta y=F(x+\delta x)+f(x+\delta x)+\phi(x+\delta x);$$

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left\{\frac{F(x+\delta x)-F(x)+f(x+\delta x)-f(x)+\phi(x+\delta x)-\phi(x)}{\delta x}\right\}$$

$$=\mathrm{Lt}_{\delta x=0}\left\{\frac{F(x+\delta x)-F(x)}{\delta x}+\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{\phi(x+\delta x)-\phi(x)}{\delta x}\right\}.$$

Now $\mathrm{Lt}_{\delta x=0}\left\{\dfrac{F(x+\delta x)-F(x)}{\delta x}+\dfrac{f(x+\delta x)-f(x)}{\delta x}+\ldots\right\}$

is equal to

$$\mathrm{Lt}_{\delta x=0}\frac{F(x+\delta x)-F(x)}{\delta x}+\mathrm{Lt}_{\delta x=0}\frac{f(x+\delta x)-f(x)}{\delta x}+\ldots,$$

because it is obvious that each term is independent of the others, since δx is put zero in each.

Also $\text{Lt}_{\delta x=0}\dfrac{F(x+\delta x)-F(x)}{\delta x}=\dfrac{dF(x)}{dx}$,

or the differential coefficient of $F(x)$

Hence, $$\frac{dy}{dx}=\frac{dF(x)}{dx}+\frac{df(x)}{dx}+\frac{d\phi(x)}{dx}.$$

We may express the result in words as follows: **The differential coefficient of the sum of a series of functions is the sum of the differential coefficients of each of the respective functions**

$dF(x)$ is often written $F'(x)$, and similarly for the others.

Ex. 1. $y=x^3+x^2$;

$$\frac{dy}{dx}=3x^2+2x$$

Ex. 2 $y=a+x+x^2+x^3+x^4$,

$$\frac{dy}{dx}=0+1+2x+3x^2+4x^3.$$

Differentiation of a function of a function.—The meaning of the term function of a function of x will be clear from the following examples ·

Ex. 3. Let $$y=\sqrt{(1+x^2)} \qquad (1)$$

This is a function of a function of x.

If we substitute a letter such as z for the quantity in the bracket, we obtain from (1)

$$y=\sqrt{z};$$

where $$z=1+x^2,$$

z is a function of x, and y is a function of z

Hence y, a function of z,—which is itself a function of x,—is said to be a function of a function of x.

Ex. 4. Similarly, if $y=\cos(x^2)$,

let $$x^2=z;$$

$$\therefore \; y=\cos z.$$

$\therefore$ y is the cosine of a function of x, and is a function of a function of x.

We can obtain in each case, with some labour, the differential coefficient of a complex function from first principles. Referring to Ex. 3, let $$y=\sqrt{(1+x^2)};$$

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{\sqrt{1+x^2+2x\delta x+(\delta x)^2}-\sqrt{(1+x^2)}}{\delta x}$$

$$=\mathrm{Lt}_{\delta x=0}\frac{(1+x^2)^{\frac{1}{2}}}{\delta x}\left[\left\{1+\frac{\delta x(2x+\delta x)}{1+x^2}\right\}^{\frac{1}{2}}-1\right]$$

By the binomial theorem,

$$\left\{1+\frac{\delta x(2x+\delta x)}{1+x^2}\right\}^{\frac{1}{2}}=1+\frac{1}{2}\frac{\delta x(2x+\delta x)}{1+x^2}-\frac{1}{4}\frac{(2x+\delta x)^2}{(1+x^2)^2}\delta x^2+\text{etc.}$$

$$\therefore \frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{(1+x^2)^{\frac{1}{2}}}{\delta x}$$

$$\times\left\{1+\frac{1}{2}\frac{\delta x(2x+\delta x)}{1+x^2}-\frac{1}{4}\left(\frac{2x+\delta x}{1+x^2}\right)^2\delta x^2+\text{etc}-1\right\}$$

$$=\mathrm{Lt}_{\delta x=0}(1+x^2)^{\frac{1}{2}}\left\{\frac{1}{2}\frac{2x+\delta x}{1+x^2}-\frac{1}{4}\left(\frac{2x+\delta x}{1+x^2}\right)^2\delta x+\quad\right\}$$

$$=\mathrm{Lt}_{\delta x=0}(1+x^2)^{\frac{1}{2}}\left[\frac{x}{1+x^2}-\left\{\frac{1}{4}\left(\frac{2x+\delta x}{1+x^2}\right)^2+\frac{1}{2(1+x^2)}\right\}\delta x\right],$$

and hence, $\quad\dfrac{dy}{dx}=(1+x^2)^{\frac{1}{2}}\dfrac{x}{1+x^2}=\dfrac{x}{(1+x^2)^{\frac{1}{2}}}$

This may be written in the form

$$\frac{dy}{dx}=\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\times 2x$$

Again referring to (3), if

$$z=1+x^2,\ \text{then}\ y=z^{\frac{1}{2}},$$

and

$$\frac{dy}{dz}=\frac{1}{2}z^{-\frac{1}{2}}=\frac{1}{2}(1+x^2)^{-\frac{1}{2}};$$

$$\frac{dz}{dx}=\frac{d}{dx}(1+x^2)=2x;$$

$$\frac{dy}{dx}=\frac{dy}{dz}\,\frac{dz}{dx}$$

Let $y=f(z)$ where $z=F(x)$, then $y=f\{F(x)\}$

If x increases to $x+\delta x$, z will increase to $z+\delta z$ where

$$z+\delta z=F(x+\delta x)$$

and

$$\delta z=F(x+\delta x)-F(x).$$

Using $z+\delta z$, we can calculate $y+\delta y$ from $y=f(z)$.

This result will be the same as if $x+\delta x$ had been substituted directly in $y=f\{F(x)\}$.

Under these conditions we can say that

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta z}\times\frac{\delta z}{\delta x},$$

because δz is the same in the ratio $\frac{\delta z}{\delta x}$ as in $\frac{\delta y}{\delta z}$. Also δy is the same in the ratio $\frac{\delta y}{\delta z}$ as in $\frac{\delta y}{\delta x}$. This will be true no matter how small δx is.

If we now assume δx to be made smaller and smaller without limit. Then

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}.$$

Thus, to calculate $\frac{dy}{dx}$ where $y=f(z)=f\{F(x)\}$, we may first find $\frac{dy}{dz}$ from $y=f(z)$, then $\frac{dz}{dx}$ from $z=F(x)$, and the product of the results is $\frac{dy}{dx}$.

Geometrical illustration.—The preceding considerations may be illustrated graphically as follows:

In Fig. 110, (i) represents $z=F(x)$, $z=x^{\frac{1}{2}}$,

(ii) " $y=f(z)$, $y=\cos z$;

(iii) " $y=f\{F(x)\}$, $y=\cos x^{\frac{1}{2}}$

Take $x=Op$ and $x+\delta x=Oq$. Draw the corresponding ordinates of (i). Measure in (ii) $Ot=Pp$, $Os=Qq$,

i.e. $Ot=z$ and $Os=z+\delta z$

Since from (i) $Pp=z$ and $Qq=z+\delta z$,

in (iii), $Or=Op=x$,

$Ov=Oq=x+\delta x$

Then $Rr=Tt=z$,

$Vv=Ss=z+\delta z$;

$\therefore\ Vl=Sm$ and $Rl=pq$.

It follows, therefore, that

$$\frac{Sm}{Tm} \times \frac{Qn}{Pn} = \frac{Sm}{Pn} = \frac{Vl}{Rl}$$

Now, if pq be made smaller and smaller without limit till it becomes zero,

$$\frac{Sm}{Tm} \text{ becomes } \frac{dy}{dz}, \text{ i.e. slope of (ii) at } T,$$

$$\frac{Qn}{Pn} \text{ becomes } \frac{dz}{dx}, \text{ i.e. slope of (i) at } P,$$

and $$\frac{Vl}{Rl} \text{ becomes } \frac{dy}{dx}, \text{ i.e slope of (iii) at } R.$$

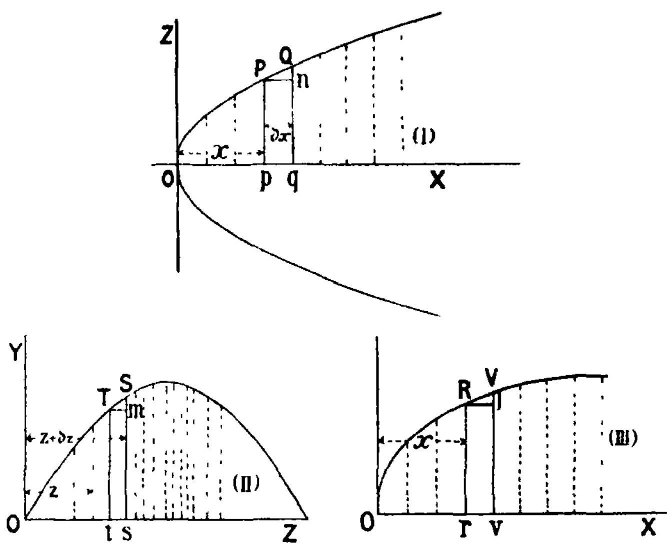

FIG 110.—To show that $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$

P, T and R being three corresponding points as described.

$$\frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dx}.$$

The relation $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$ is of great use in complicated expressions.

Ex. 5. Thus, if $y=(x+x^2)^2$. . . ..(1)
Let $z=x+x^2$, then (1) becomes $y=z^2$.

Then $$\frac{dz}{dx}=1+2x;$$

also $$\frac{dy}{dz}=2z,$$

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=2z(1+2x)$$
$$=2(x+x^2)(1+2x).$$

Ex. 6 Let $$y=\sqrt{(a^2-x^2)}.$$
Assume $$z=a^2-x^2$$
Then $$y=z^{\frac{1}{2}},$$
$$\frac{dy}{dz}=\frac{1}{2}z^{-\frac{1}{2}};$$

also, $$\frac{dz}{dx}=-2x;$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}\times(-2x)$$
$$=-x(a^2-x^2)^{-\frac{1}{2}}$$

Ex. 7. When the temperature of platinum wire is increased, the variation of electrical resistance, with temperature t, is given by

$$R=R_0(1+\alpha t+\beta t^2) \quad . \quad . \qquad (1)$$

The increase in the resistance is given by the differential coefficient of (1) multiplied by the small rise in temperature;

$$\frac{dR}{dt}=R_0(\alpha+2\beta t).$$

Ex 8. Find $\frac{dy}{dx}$ when $y=\sin x^2$.

Put $$z=x^2; \quad \therefore \frac{dz}{dx}=2x.$$

$$y=\sin z; \quad \therefore \frac{dy}{dz}=\cos z.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\cos z\times 2x$$
$$=2x\cos x^2.$$

Ex. 9. Find $\frac{dy}{dx}$ when $y=(x^2+4)^4$.

Put $$z=x^2+4, \qquad \frac{dz}{dx}=2x$$

$$y=z^4; \therefore \frac{dy}{dz}=4z^3.$$

Hence, $$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=4z^3\times 2x$$
$$=8x(x^2+4)^3.$$

Ex 10. Find $\frac{dx}{dy}$ when $y=\frac{1}{x^2+x+c}$.

Let $$z=x^2+x+c, \therefore \frac{dz}{dx}=2x+1,$$

and $$y=\frac{1}{z}; \qquad \frac{dy}{dz}=-z^{-2}.$$

$$\therefore \frac{dy}{dx}=-\frac{2x+1}{(x^2+x+c)^2}$$

Ex. 11. If x increases uniformly at the rate of 0·001 ft. per sec., at what rate is the expression $(1+x)^3$ increasing per second, when x becomes 9?

Let $z=1+x$, then $y=z^3$,

$$\frac{dz}{dx}=1 \text{ and } \frac{dy}{dz}=3z^2$$

Substituting, $$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=3(1+x)^2.$$

When x becomes 9 this gives 300, or y increases 300 times as quickly as x.

But x increases 0·001 ft. per sec;

$\therefore$ y increases at $300\times 0·001=0·3$ ft. per sec.

Differential coefficient of the product of two functions.

Ex. 1. Let $$y=x^2\cos x.$$

This is a typical representative of a large family of functions. Its differential coefficient may be found by either of the following methods.

$$\frac{dy}{dx} = \mathrm{Lt}_{\delta x=0}\left[\frac{(x+\delta x)^2\cos(x+\delta x) - x^2\cos x}{\delta x}\right]$$

$$= \mathrm{Lt}_{\delta x=0}\left[\frac{x^2\{\cos(x+\delta x) - \cos x\} + 2x\delta x\cos(x+\delta x) + (\delta x)^2\cos(x+\delta x)}{\delta x}\right]$$

$$= \mathrm{Lt}_{\delta x=0}\left[\frac{x^2\{\cos(x+\delta x) - \cos x\}}{\delta x} + 2x\cos(x+\delta x) + \delta x\cos(x+\delta x)\right]$$

$$= \mathrm{Lt}_{\delta x=0}\left[\frac{x^2 \times 2\sin\left(x+\frac{\delta x}{2}\right) \times \left(-\sin\frac{\delta x}{2}\right)}{\delta x} + 2x\cos(x+\delta x) + \delta x\cos(x+\delta x)\right];$$

$$\therefore \frac{dy}{dx} = -x^2\sin x + 2x\cos x$$

Instead of the preceding method of solution, the result could be obtained as follows

$$\frac{dy}{dx} = \mathrm{Lt}_{\delta x=0}\left[\frac{(x+\delta x)^2\cos(x+\delta x) - x^2\cos x}{\delta x}\right]$$

$$= \mathrm{Lt}_{\delta x=0}\left[\frac{(x+\delta x)^2\cos(x+\delta x) - (x+\delta x)^2\cos x + (x+\delta x)^2\cos x - x^2\cos x}{\delta x}\right]$$

$(x+\delta x)^2\cos x$ has been added and subtracted in the numerator, then, by rearrangement of the terms, we obtain

$$\frac{dy}{dx} = \mathrm{Lt}_{\delta x=0}\left\{\frac{(x+\delta x)^2\{\cos(x+\delta x) - \cos x\}}{\delta x} + \frac{(x+\delta x)^2 - x^2}{\delta x}\cos x\right\}$$

But we have already found that

$$\mathrm{Lt}_{\delta x=0}\left\{\frac{\cos(x+\delta x) - \cos x}{\delta x}\right\}$$

is the differential coefficient of $\cos x$, or $\frac{d}{dx}(\cos x)$

Similarly, $$\mathrm{Lt}_{\delta x=0}\left\{\frac{(x+\delta x)^2 - x^2}{\delta x}\right\}$$

is the differential coefficient of x^2, or $\frac{d}{dx}(x^2)$.

Now, in the limit, $(x+\delta x)^2$ is x^2;

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2)$$

$$= -x^2\sin x + 2x\cos x.$$

These operations apply to any case, and the following proof is only a repetition, using symbols instead of the preceding concrete case. Comparison should be made step by step

Thus, instead of x^2 and $\cos x$, write $f(x)$ and $F(x)$, respectively.

Then, $$y=f(x)\times F(x).$$

Hence, $$y+\delta y=f(x+\delta x)\times F(x+\delta x),$$

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left[\frac{f(x+\delta x)\times F(x+\delta x)-f(x)\times F(x)}{\delta x}\right].$$

This may be written in the form

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left[\frac{\begin{array}{c}f(x+\delta x)\times F(x+\delta x)-f(x+\delta x)\times F(x)\\ +f(x+\delta x)\times F(x)-f(x)\times F(x)\end{array}}{\delta x}\right]$$

$$=\mathrm{Lt}_{\delta x=0}\left[f(x+\delta x)\frac{F(x+\delta x)-F(x)}{\delta x}+\frac{f(x+\delta x)-f(x)}{\delta x}F(x)\right]$$

But $$\mathrm{Lt}_{\delta x=0}\left\{\frac{F(x+\delta x)-F(x)}{\delta x}\right\} \text{ is } \frac{d}{dx}F(x),$$

i.e. the differential coefficient of $F(x)$ with respect to x.

Also $$\mathrm{Lt}_{\delta x=0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\} \text{ is } \frac{d}{dx}f(x).$$

Similarly, $f(x+\delta x)$ becomes $f(x)$

Hence $$\frac{dy}{dx}=f(x)\frac{d}{dx}F(x)+F(x)\frac{d}{dx}f(x).$$

The following demonstration is very general, and perhaps better for comparison with the example

Let $y=u\times v$, where u and v are functions of x

When x increases to $x+\delta x$, y becomes $y+\delta y$, u becomes $u+\delta u$, and v becomes $v+\delta v$;

$$y+\delta y=(u+\delta u)(v+\delta v),$$

and $$\frac{\delta y}{\delta x}=\frac{(u+\delta u)(v+\delta v)-uv}{\delta x}$$

$$=u\frac{\delta v}{\delta x}+\left(v\frac{\delta u}{\delta x}\right)+\frac{\delta u\delta v}{\delta x}.$$

Now, as δx becomes smaller and smaller,
$\frac{\delta v}{\delta x}$ approaches nearer and nearer to $\frac{dv}{dx}$, $\frac{\delta u}{\delta x}$ to $\frac{du}{dx}$, $\frac{\delta y}{\delta x}$ to $\frac{dy}{dx}$, and $\frac{\delta u \,.\, \delta v}{\delta x}$ becomes 0.

Hence, in the limit,

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx} \quad \ldots\ldots \quad \ldots\ldots \quad \ldots \quad \text{(i)}$$

The preceding important result may be stated in words as follows:—**The differential coefficient of the product of two functions is the sum of the products of each function by the differential coefficient of the other.**

As a first use of this theorem consider

$$y=\text{const} \times f(x).$$

Then $$\frac{dy}{dx}=\text{const} \times \frac{df(x)}{dx}+f(x)\frac{d(\text{const})}{dx}$$

But the differential coefficient of a constant is zero;

$$\frac{dy}{dx}=\text{const} \times \frac{df(x)}{dx},$$

or is simply the product of the same constant and the differential coefficient of the function. Simple examples which may easily be verified may be manufactured as follows

Ex. 2 Let $$y=20x^6,$$

$$\frac{dy}{dx}=120x^5.$$

As $20x^6=4x^4 \times 5x^2$, we can also obtain the result from (i) as follows.

$$\frac{dy}{dx}=4x^4 \times 10x+5x^2 \times 16x^3=120x^5$$

In a similar manner, when $y=uvw$,

$$\frac{dy}{dx}=uv\frac{dw}{dx}+vw\frac{du}{dx}+uw\frac{dv}{dx}$$

To obtain familiarity with the method it may be advisable, as in the preceding case, to select some fairly easy example and proceed to apply the rule to it.

Ex. 3. Let $y = 24x^9 = 2x^2 \times 3x^3 \times 4x^4$,

$$\frac{dy}{dx} = 2x^2 \times 3x^3 \times \frac{d(4x^4)}{dx} + 2x^2 \times 4x^4 \times \frac{d(3x^3)}{dx} + 3x^3 \times 4x^4 \times \frac{d(2x^2)}{dx}$$

$$= 96x^8 + 72x^8 + 48x^8 = 216x^8,$$

and this can be verified readily, because if $y = 24x^9$,

$$\therefore \frac{dy}{dx} = 9 \times 24x^8 = 216x^8.$$

Ex. 4. A rectangular slab of wrought iron is heated and its linear dimensions increase at the rate 0·01 inch per sec. Find the rate at which its volume is increasing at the instant when the dimensions are 4, 3, and 12 inches respectively.

If $y = uvw$, where u, v, and w are functions of t, the time denoting three edges of the solid mutually at right angles, then

$$\frac{dy}{dt} = vw\frac{du}{dt} + uw\frac{dv}{dt} + uv\frac{dw}{dt}. \qquad \text{(ii)}$$

But y denotes the volume of the solid, and $\therefore \frac{dy}{dt}$ denotes the rate of increase of volume due to change of temperature.

Hence, at the instant when the three dimensions are 4, 3, and 12, the rate of increase of the volume is obtained from (ii) by substituting the given values, and is

$$(36 \times 0{\cdot}01) + (48 \times 0{\cdot}01) + (12 \times 0{\cdot}01) = 96 \times 0{\cdot}01;$$

$$\therefore \frac{dV}{dt} = 0{\cdot}96 \text{ cub. in per sec.}$$

Ex 5. Find $\frac{dy}{dx}$ when $y = (x^3 + a)(3x^2 + b)$.

$$\frac{dy}{dx} = (x^3 + a)\frac{d(3x^2 + b)}{dx} + (3x^2 + b)\frac{d(x^3 + a)}{dx}$$

$$= (x^3 + a)6x + (3x^2 + b)3x^2$$

$$= 15x^4 + 3bx^2 + 6ax.$$

Ex. 6 Find $\frac{dy}{dx}$ when $y = (a + x)(b + x)(c + x)$.

$$\frac{dy}{dx} = (b + x)(c + x)\frac{d(a + x)}{dx} + \ldots$$

$$= 3x^2 + 2(a + b + c)x + ab + ac + bc$$

Ex. 7. Find $\frac{dy}{dx}$ when $y=a(bx^2)^4$.

Let $z=bx^2$.

Then $y=az^4$,

$$\frac{dy}{dz}=4az^3 \text{ and } \frac{dz}{dx}=2bx;$$

$$\frac{dy}{dx}=4az^3\times 2bx=4a(bx^2)^3\times 2bx$$

$$=8abx(bx^2)^3=8ab^4x^7.$$

EXERCISES. XXXVI.

Find in each of the following cases the value of $\frac{dy}{dx}$; verify the result obtained by calculation from first principles.

1. $y=7x^2$.
2. $y=3\sin x$.
3. $y=\cos 3x$.
4. $y=5\cos(2x+3)$
5. $y=\log 6x$.
6. $y=A\log x^3$.
7. $y=3e^{2x}$.
8. $y=Ae^{-kx}$.

Find the values of $\frac{ds}{dt}$ in the following examples:

9. $s=3t^2-4t+7$.
10. $s=At^2+Bt+c$.
11. $s=3\sin(4t+9)$.
12. $s=7\cos^2(6t^3+9t+5)$.
13. $s=14e^{\frac{t}{8}}+9\sin 8t$.
14. $s=11e^t\sin(6t+7)$.
15. $s=Ae^{bt}\sin(ct+f)$.

Quotient of two functions.—To obtain a general expression for the differentiation of the quotient of two functions we may proceed as follows:

Let $y=\frac{f(x)}{F(x)}$,

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left[\frac{\frac{f(x+\delta x)}{F(x+\delta x)}-\frac{f(x)}{F(x)}}{\delta x}\right]$$

$$=\mathrm{Lt}_{\delta x=0}\left[\frac{F(x)f(x+\delta x)-f(x)F(x+\delta x)}{F(x)F(x+\delta x)\delta x}\right];$$

therefore,

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left[\frac{F(x)f(x+\delta x)-F(x)f(x)+F(x)f(x)-f(x)F(x+\delta x)}{F(x)F(x+\delta x)\delta x}\right].$$

In the numerator $f(x)F(x)$ has been added and subtracted; this allows $\frac{dy}{dx}$ to be put into the following form:

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left[\frac{F(x)\frac{f(x+\delta x)-f(x)}{\delta x}-f(x)\frac{F(x+\delta x)-F(x)}{\delta x}}{F(x)F(x+\delta x)}\right],$$

and finally, taking the limiting values of the functions in the numerator and denominator,

$$\frac{dy}{dx}=\frac{F(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}F(x)}{\{F(x)\}^2}$$

Alternative proof.—An alternate form of proof of the preceding result may be obtained.

Thus, let $y=\frac{u}{v}$,

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left[\frac{\frac{u+\delta u}{v+\delta v}-\frac{u}{v}}{\delta x}\right]$$

$$=\mathrm{Lt}_{\delta x=0}\left[\frac{v\frac{\delta u}{\delta x}-u\frac{\delta v}{\delta x}}{v(v+\delta v)}\right].$$

Hence,
$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

Or, **the differential coefficient of a quotient of two functions is the product of the denominator and the differential coefficient of the numerator, minus the product of the numerator and the differential coefficient of the denominator, divided by the denominator squared.** This important rule may be tested as follows.

Ex 1. $y=\frac{10x^6}{2x^2}$, y is really $5x^4$, but consider it as a quotient.

Then
$$\frac{dy}{dx}=\frac{2x^2\frac{d}{dx}(10x^6)-10x^6\frac{d}{dx}(2x^2)}{(2x^2)^2}$$

$$=\frac{2x^2\times 60x^5-40x^7}{4x^4}=20x^3.$$

As $y=5x^4$, we see that $\frac{dy}{dx}=20x^3$.

Ex. 2. $y=\tan x$, find $\dfrac{dy}{dx}$.

By our rule, since $y=\dfrac{\sin x}{\cos x}$,

$$\frac{dy}{dx}=\frac{\cos x\dfrac{d}{dx}(\sin x)-\sin x\dfrac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

From first principles,

$$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\left[\frac{\tan(x+\delta x)-\tan x}{\delta x}\right]$$

$$=\mathrm{Lt}_{\delta x=0}\left[\frac{\sin(x+\delta x)\cos x-\sin x\cos(x+\delta x)}{\delta x\,.\,\cos x\cos(x+\delta x)}\right]$$

$$=\mathrm{Lt}_{\delta x=0}\left[\frac{\sin\{(x+\delta x)-x\}}{\delta x\cos x\cos(x+\delta x)}\right]$$

$$=\mathrm{Lt}_{\delta x=0}\left[\frac{\sin\delta x}{\delta x}\quad\frac{1}{\cos x\cos(x+\delta x)}\right]$$

In the limit, when $\delta x=0$, $\left[\dfrac{\sin\delta x}{\delta x}\right]=1$ (p. 383),

$$\frac{dy}{dx}=\frac{1}{\cos^2 x}=\sec^2 x.$$

Differentiation of inverse functions.—We proceed to prove

that $$\frac{dy}{dx}\times\frac{dx}{dy}=1$$

By definition $$\frac{dy}{dx}=\mathrm{Lt}_{\delta x=0}\frac{\delta y}{\delta x},$$

and $$\frac{dx}{dy}=\mathrm{Lt}_{\delta y=0}\frac{\delta x}{\delta y},$$

therefore, $$\frac{dy}{dx}\cdot\frac{dx}{dy}=\mathrm{Lt}_{\delta x=0}\frac{\delta y}{\delta x}\times\mathrm{Lt}_{\Delta y=0}\frac{\Delta x}{\Delta y}.$$

Now the product of the limiting values of two or more functions is equal to the limit of the products, and therefore

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=\mathrm{Lt}_{\substack{\delta x=0\\ \Delta y=0}}\left[\frac{\delta y}{\delta x}\times\frac{\Delta x}{\Delta y}\right].$$

Before the limit is taken, δy and δx are of any value corresponding to each other, as are also Δx and Δy, and, as we have seen previously, the limit is independent of such quantities. Since this is the case, make $\Delta y = \delta y$, and then Δx will $= \delta x$, and we have

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = \mathrm{Lt}_{\substack{\delta x = 0 \\ \delta y = 0}} \left[\frac{\delta y}{\delta x} \times \frac{\delta x}{\delta y} \right] = 1.$$

Ex. 1. $y = x^3$,

then $x = y^{\frac{1}{3}}$,

$$\therefore \frac{dy}{dx} = 3x^2,$$

and $\frac{dx}{dy} = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{x^2}$;

$$\therefore \frac{dy}{dx} \times \frac{dx}{dy} = 3x^2 \times \frac{1}{3x^2} = 1.$$

Ex. 2. $y = x^2$;

$$\therefore x = \pm y^{\frac{1}{2}},$$

$$\frac{dy}{dx} = 2x \text{ and } \frac{dx}{dy} = \pm \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2x},$$

where the $\pm$ signs agree with those before;

$$\frac{dy}{dx} \times \frac{dx}{dy} = 2x \times \frac{1}{2x} = 1.$$

Geometrical proof.—A geometrical proof that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ may be obtained as follows:

Let $Q'PQ$ (Fig. 111) be a portion of a curve representing

$$y = f(x).$$

Then, as on p 305,

$$\frac{dy}{dx} = \mathrm{Lt}_{\delta x = 0} \frac{QM}{PM} \quad \text{(Fig. 111)}$$

$$= \tan \theta.$$

Again, $\frac{dx}{dy} = \mathrm{Lt}_{\Delta y = 0} \frac{\Delta x}{\Delta y}.$

Now, as Δy gets less and less, Q' must get nearer to point P, and eventually PQ' will coincide with the tangent at P, and the angle ϕ will become equal to θ.

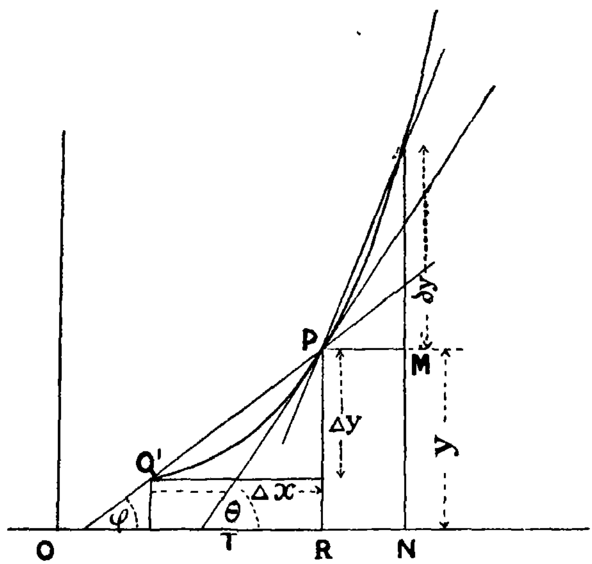

FIG. 111 —To show that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$

But $\frac{\Delta x}{\Delta y} = \cot \phi$, and, therefore, in the limit, when ϕ becomes θ,

$$\mathrm{Lt}_{\Delta y=0}\left[\frac{\Delta x}{\Delta y}\right] \text{ becomes } \cot \theta\,;$$

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = \mathrm{Lt}_{\delta x=0}\left[\frac{PM}{QM}\right] \times \mathrm{Lt}_{\delta x=0}\left[\frac{-\Delta x}{-\Delta y}\right]$$

$$= \tan \theta \times \cot \theta = 1.$$

The theorem that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ is very useful in finding the rates of increase, or differential coefficients, of certain functions as follows:

Ex. 3. Let $$y=\sin^{-1}\frac{x}{a}.$$

Since $$y=\sin^{-1}\frac{x}{a},\quad \frac{x}{a}=\sin y,$$

$$\frac{dx}{dy}=a\cos y=a\sqrt{1-\sin^2 y}\,;$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{a^2-x^2}}.$$

Ex 4 Let $y=\sin^{-1}x$, a particular case of the preceding example,

then $$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}.$$

Similarly, if $$y=\cos^{-1}\frac{x}{a},$$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{a^2-x^2}}$$

Thus if $y_1=\sin^{-1}x$ and $y_2=\cos^{-1}x$; y_1+y_2 has for its least value $\frac{1}{2}\pi\left(\text{but in any case is constant}=\frac{4n+1}{2}\pi\right)$;

$$\frac{dy_1}{dx}+\frac{dy_2}{dx}=\frac{d(\frac{1}{2}\pi)}{dx}=0$$

Hence $$\frac{dy_1}{dx}=-\frac{dy_2}{dx};\quad \therefore\quad \frac{d(\sin^{-1}x)}{dx}=-\frac{d(\cos^{-1}x)}{dx},$$

so $$\frac{d(\tan^{-1}x)}{dx}=-\frac{d(\cot^{-1}x)}{dx}.$$

Ex. 5. Let $$y=\cos^{-1}\frac{x}{a};$$

$$a\cos y=x$$

$$-a\sin y\frac{dy}{dx}=1\,;$$

$$\therefore\ \frac{dy}{dx}=-\frac{1}{a\sqrt{1-\cos^2 y}}=-\frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}$$

$$=-\frac{1}{\sqrt{a^2-x^2}}.$$

Ex. 6. Let $$y=\tan^{-1}\frac{x}{a}.$$

Here
$$x=a\tan y;$$
$$\therefore \frac{dx}{dy}=a\sec^2 y=a(1+\tan^2 y)$$
$$=a\left\{1+\left(\frac{x}{a}\right)^2\right\};$$
$$\therefore \frac{dy}{dx}=\frac{1}{a\left(1+\frac{x^2}{a^2}\right)}=\frac{a}{a^2+x^2}.$$

Ex. 7. Similarly, if $y=\cot^{-1}\frac{x}{a}$,

then
$$\frac{dy}{dx}=-\frac{a}{x^2+a^2}.$$

These cases of inverse functions, viz
$$\sin^{-1}\frac{x}{a},\ \cos^{-1}\frac{x}{a},\ \tan^{-1}\frac{x}{a},\ \cot^{-1}\frac{x}{a},$$
are of great importance in the application of mathematics to physical and mechanical sciences.

Ex. 8. Let
$$y=x^{\frac{1}{m}}; \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$
$$\therefore x=y^m,$$
$$\frac{dx}{dy}=my^{m-1};$$
$$\therefore \frac{dy}{dx}=\frac{1}{my^{m-1}}. \quad \ldots\ldots\ldots \quad \text{(ii)}$$

Substitute in (ii) the value of y from (i);
$$\therefore \frac{dy}{dx}=\frac{1}{m\left(x^{\frac{1}{m}}\right)^{m-1}}=\frac{1}{m}x^{\left(\frac{1}{m}-1\right)}.$$

Ex. 9. If the diameter of a circle increases at the rate of 0·01 inch per second, at what rate is the area increasing when the initial diameter is 10 inches?

Here, if x denote the diameter, and $x+\delta x$ the increase in length;

then
$$\text{area}=y=\frac{\pi}{4}x^2, \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(i)}$$

and
$$y+\delta y=\frac{\pi}{4}(x+\delta x)^2=\frac{\pi}{4}\{x^2+2x\delta x+(\delta x)^2\}. \quad \ldots\ldots \quad \text{(ii)}$$

Subtracting (i) from (ii) and dividing by δx,

$$\frac{\delta y}{\delta x}=\frac{\pi}{4}\times 2x+\frac{\pi}{4}\delta x$$

Hence, average rate of increase when $x=10$ is given by

$$\frac{\delta y}{\delta x}=0{\cdot}7854\times 20+0{\cdot}7854\times \delta x.$$

It will be seen that the second term on the right hand side becomes smaller and smaller as δx is diminished Finally, when δx is indefinitely small, the actual rate of increase,

or $$\frac{dy}{dx}=15{\cdot}708.$$

That is, the area changes 15·708 times as quickly as the radius at this point, or is increasing $15{\cdot}708\times 0{\cdot}01$ sq in. per sec. $=0{\cdot}15708$ sq. in. per sec.

Ex. 10. If the diameter of a spherical soap-bubble increases uniformly at the rate of 0 1 inch per second, at what rate is the volume increasing when the diameter is 3 inches?

Let V denote the volume and x the diameter.

Then $$V=\frac{\pi}{6}x^3, \qquad (i)$$

also $$V+\delta V=\frac{\pi}{6}(x+\delta x)^3=\frac{\pi}{6}\{x^3+3x^2\delta x+3x(\delta x)^2+(\delta x)^3\}. \qquad (ii)$$

Subtracting (i) from (ii) and dividing by δx,

$$\frac{\delta V}{\delta x}=\frac{\pi}{6}\{3x^2+3x\delta x+(\delta x)^2\}$$

When x is 3, we obtain for average rate of increase

$$\frac{\delta V}{\delta x}=0{\cdot}5236\{27+9\times \delta x+(\delta x^2)\}.$$

When δx is indefinitely small,

$$\frac{dV}{dr}=0{\cdot}5236\times 27=14{\cdot}137\,;$$

$\therefore$ rate of increase of volume is $14{\cdot}137\times 0{\cdot}1=1{\cdot}4137$ cubic inches per second.

Ex. 11. If the radius of a soap-bubble is increasing at the rate of 0·05 inch per second, at what rate is the capacity increasing when the radius becomes one inch?

V=volume of a sphere$=\frac{4}{3}\pi r^3$, where r denotes the radius of the sphere;

$$\therefore \quad \frac{dV}{dr}=4\pi r^2;$$

$$\therefore \quad \delta V=4\pi r^2\delta r, \text{ when } \delta r \text{ is small}, =4\pi\times 1^2\times 0\cdot 05$$
$$=0\cdot 2\pi \text{ cub. in per sec.} \qquad =0\cdot 6283 \text{ cub. in per sec.}$$

Tangent, subtangent, and subnormal.—Let P (Fig. 112) be a point on the curve $y=f(x)$, the coordinates of the point P being $OM=x$ and $MP=y$.

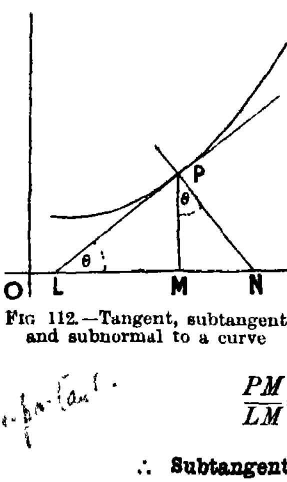

FIG 112.—Tangent, subtangent, and subnormal to a curve

If L be the point where the tangent at P cuts the axis of x, and if PN is a line perpendicular to PL and meeting the axis of x at N, then LP is the **tangent**, PN is the **normal**, LM is the **subtangent**, and MN the **subnormal** to the curve at P.

If θ denotes the angle which the tangent makes with the axis of x, then the angle $PNM=\frac{\pi}{2}-\theta$.

$$\frac{PM}{LM}=\tan\theta=\frac{dy}{dx}.$$

$$\therefore \quad \textbf{Subtangent}=LM=y\div\frac{dy}{dx}=\mathbf{y}\,\frac{\mathbf{dx}}{\mathbf{dy}} \qquad \text{(i)}$$

Also
$$\frac{MN}{PM}=\tan\theta=\frac{dy}{dx};$$

$$\textbf{subnormal}=MN=\mathbf{y}\,\frac{\mathbf{dy}}{\mathbf{dx}} \qquad \text{(ii)}$$

The lengths of the normal PN and tangent PL are easily obtained.

Thus,
$$PN=\sqrt{PM^2+MN^2}=\sqrt{y^2+y^2\left(\frac{dy}{dx}\right)^2};$$

$$\therefore \quad \textbf{normal}=PN=\mathbf{y}\sqrt{\mathbf{1}+\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2}.$$

Similarly, **tangent** $=PL=\sqrt{PM^2+ML^2}=\dfrac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\dfrac{dy}{dx}}.$

Ex. 1. Draw a tangent and normal at a given point P on the curve $y^2=4ax$. Plotting on squared paper, a curve called a parabola is obtained as in Fig. 113.

Differentiating, we obtain

$$2y\frac{dy}{dx}=4a\,;$$

$$\frac{dy}{dx}=\frac{4a}{2y}=\frac{2a}{y}.$$

From (ii),

$$\text{subnormal}=y\frac{dy}{dx}=2a.$$

From (i),

$$\text{subtangent}=y\frac{dx}{dy}=\frac{y^2}{2a}=2x.$$

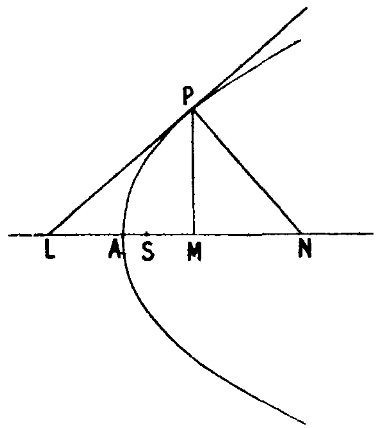

FIG. 113.—Tangent and normal to a parabola

To draw the tangent at P, make $ML=2AM$ (Fig. 113) and join P to L, then PL is the tangent at P.

To draw the normal, make $MN=2AS$ and join NP, then PN is the normal required.

Length of curve.—Let A and B be two points near together on the curve NAM (Fig. 114).

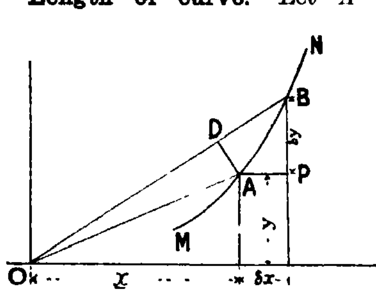

FIG. 114.—Length of a curve.

Let the coordinates of point A be denoted by (x, y); and of B by $(x+\delta x,\ y+\delta y)$.

If s denote length of curve, then AB will be represented by δs.

As AB is a very small length of curve, we may assume it to form the hypotenuse of a right-angled triangle, of which PA and PB are the two perpendicular sides.

Since $AP=\delta x$ and PB is δy,

$$(\delta s)^2=(\delta x)^2+(\delta y)^2\ ;$$

dividing by $(\delta x)^2$,

$$\left(\frac{\delta s}{\delta x}\right)^2=1+\left(\frac{\delta y}{\delta x}\right)^2\ ;$$

$$\therefore\ \frac{\delta s}{\delta x}=\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}.$$

In the limit, when δx and therefore δy are indefinitely small, we obtain

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ \ldots\ldots\ldots\ \ \ \text{(i)}$$

$$=\sqrt{1+\tan^2\phi}=\sec\phi,$$

where ϕ is the inclination of the tangent to the axis of x.

The preceding result is often required in polar coordinates.

Join the origin O to A and B (Fig. 114). Draw AD perpendicular to OB. Then, if $OA=r$, $OB=OD+DB$, we may denote DB by δr, and angle AOD by $\delta\theta$.

Now AD is very nearly the arc of a circle, whose radius is r, and which subtends an angle $\delta\theta$ at the centre of the circle this gives.

$AD=r\delta\theta$, whence we obtain from the right-angled triangle ADB,

$$(\delta s)^2=(r\delta\theta)^2+(\delta r)^2\ ;$$

or taking the square root and dividing by $\delta\theta$,

$$\frac{\delta s}{\delta\theta}=\sqrt{r^2+\left(\frac{\delta r}{\delta\theta}\right)^2}\ ;$$

$$\therefore\ \text{in the limit}\ \frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.$$

Radius of curvature.—The radius of curvature of a curve at any point is the radius of that circle which agrees most nearly with the curve at that point; also, the curvature of an arc of a circle is the reciprocal of its radius. If three points ABC be taken near together on a curve DE (Fig. 115), then a circle can be drawn through the three points, as the distance between the points is diminished, the circle will more and more nearly coincide with the curve; or, the circle drawn

through three points, indefinitely near each other, gives the radius and centre of the circle of curvature at the point.

The slope of the line passing through the two points A and B is written $\frac{\delta y}{\delta x}$; the change in $\frac{\delta y}{\delta x}$, in passing from B to C, is the change in the angle itself multiplied by $\sec^2\phi$. This increase in $\frac{\delta y}{\delta x}$ divided by the length of arc BC is therefore the average curvature from B to C, i.e. $\frac{d\phi}{ds}=\frac{1}{\rho}$.

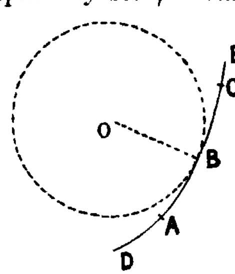

FIG. 115.—Radius of curvature

Let ρ denote radius of curvature at B.

Now write $\quad u=\tan\phi=\frac{dy}{dx}$;

and consider u, ϕ, $\frac{du}{dx}$ to be functions of x. Take the differential coefficient of this equation;

$$\frac{du}{dx}=\sec^2\phi\frac{d\phi}{dx}=\frac{d\left(\frac{dy}{dx}\right)}{dx}, \qquad \ldots\ldots\ldots\ldots\text{(i)}$$

the latter is abbreviated into $\frac{d^2y}{dx^2}$.

To obtain $\frac{d\phi}{dx}$ we use the relation

$$\frac{d\phi}{dx}=\frac{d\phi}{ds}\cdot\frac{ds}{dx}=\frac{1}{\rho}\,\frac{1}{\cos\phi}.$$

Substituting this value in (i), we obtain

$$\frac{1}{\rho}\sec^3\phi=\frac{d^2y}{dx^2},$$

also $\qquad \sec^2\phi=1+\tan^2\phi=1+\left(\frac{dy}{dx}\right)^2;$

$$\therefore\ \rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \qquad \ldots\ldots\ldots\text{(ii)}$$

Ex. 1. Find the radius of curvature at the point $x=0\cdot6$ on the curve $y=2x^3$.

As $y=2x^3$, $\frac{dy}{dx}=6x^2=2\cdot16$ when $x=0\cdot6$.

$$\frac{d^2y}{dx^2}=12x=12\times0\cdot6=7\cdot2;$$

$$\therefore \quad \rho=\frac{\{1+(2\cdot16)^2\}^{\frac{3}{2}}}{7\cdot2}=1\cdot874.$$

Ex. 2. In the parabola $y=ax^2$, find the radius of curvature at the vertex

Here $y=ax^2$ $\quad \frac{dy}{dx}=2ax$ and $\frac{d^2y}{dx^2}=2a$;

$$\frac{1}{\rho}=\frac{2a}{(1+4a^2x^2)^{\frac{3}{2}}};$$

$$\therefore \quad \rho=\frac{1}{2a} \text{ when } x=0.$$

When, as often occurs in engineering problems, the curve is a very flat one and nearly parallel to the axis of x, then the length δs may be taken to be simply the change in x. The approximation being closer as the curve is flatter; when δs becomes indefinitely small we may denote the curvature by the change in

$$\frac{dy}{dx}, \text{ i.e } \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}.$$

Hence, instead of the more accurate expression given by Eq. (ii) we can use—especially in problems dealing with beams—the approximate expression $\frac{1}{\rho}=\frac{d^2y}{dx^2}$ A result which could be obtained by putting $\frac{dy}{dx}=0$ in (ii).

EXERCISES. XXXVII.

Differentiate the following with regard to x:

1 If $y=\frac{20x^6}{2x^2}$.

2 $y=\frac{2x^4}{a^2-x^2}$

3. $\frac{x^2}{(a+x^3)^2}$.

4. $\tan x$

5. $\frac{1-x}{\sqrt{1+x^2}}$

6. (i) $\frac{mx+n}{px+q}$, (ii) $\frac{1}{x^n}$.

7. (i) $y=\dfrac{1-x}{\sqrt{1+x^2}}$, (ii) $y=x^a \log x$.

8. $y=\dfrac{(x+1)^2}{x^2+1}$.

9. $y=e^{\sin x}$.

10. $y=\log_a \sin^{-1} x$

11 $\cos\sqrt{x^2+a^2}$.

12. $\sin\sqrt{x^2+a^2}$

13 $\log\sqrt{x^2+a^2}$.

14. $y=\sin^{-1} x^2$

15. If $u=(x+1)(x^2+1)$, find $\dfrac{du}{dx}$.

16. $\tan^{-1}\dfrac{2x}{1-x^2}$

17. $\log\sqrt[4]{\dfrac{1+x}{1-x}}+\frac{1}{2}\tan^{-1}x$.

18. $\log\dfrac{x-a}{x+a}$.

19 $\dfrac{\cos 3x+\cos x}{\sin 3x-\sin x}$.

Find $\dfrac{du}{dx}$.

20. $u=\log_e(x+\sqrt{a^2+x^2})$.

21 $u=\tan^{-1}\dfrac{2x}{1-x^2}$.

22 $u=(a^2-x^2)^{\frac{3}{2}}$

23. $u=\log_e\dfrac{x^2+\sqrt{x^2-1}}{x^2-\sqrt{x^2-1}}$

24. $u=\dfrac{\sin mx}{(\cos x)^m}$.

25. $u=\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$.

26. $(a+bx)x^3$.

27. $\sqrt{x^2+a^2}$.

28. $\sin^2 x$.

29 $\sin^3 x\cos x$

30 $(ax+x^2)^2$.

31. $e^x\cos x$.

32. $x^c\log_a x$.

33. $x\sqrt{(a^2-x^2)}$.

34 $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$.

35. $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$.

36. If y is the area of a circle of radius x, show that

$$\frac{dy}{dx}=2\pi x$$

37. If y denotes the surface of a sphere of radius x,

$$\frac{dy}{dx}=8\pi x.$$

Show also that $\dfrac{dy}{dx}=4\pi x^2$, where y denotes the volume.

38. The volume of a spherical balloon is increasing at the rate of c cubic feet per second when the diameter is x feet. What is the rate of increase of the superficial area of the balloon at that instant?

CHAPTER XVI.

RATES OF INCREASE. VELOCITY. ACCELERATION AND FORCE.

Rates of increase.—Probably everyone is more or less familiar with the statement that the average speed, or velocity, of a train is 50 miles per hour. Thus, suppose a train takes 8 hours for a journey of 400 miles, then to obtain the average speed, the number denoting the distance is divided by the number denoting the time. Or, more shortly, the distance divided by the time gives an average speed of 50 miles per hour. But, during the 8 hours the train has many times reduced its speed, stopped altogether, and increased its speed again, so that the average rate of 50 miles an hour gives no measure of its speed at any given instant, such as when passing a station on the line of route. How can we proceed to measure the speed of the train when passing such a place? We might perhaps set out a distance of 176 yards, close to the station, and measure as accurately as possible the time, say 6 seconds or $\frac{1}{600}$ hour, which a given point in the train takes to pass over the distance: then, the distance divided by the time $\frac{1}{10} \div \frac{1}{600} = 60$ miles per hour, gives us the average speed or velocity of the train during 6 seconds while passing over 176 yards.

If instead of 176 yards we use the symbol δs, and instead of 6 seconds the symbol δt, then we have the average speed for this interval of time expressed by $\frac{\delta s}{\delta t}$. Now, as δt, and therefore δs, get smaller and smaller, this result gets more and more nearly equal to the actual velocity of the train at

the station. But the distance and time may be made so small that we have no means of measuring them. It would therefore be impossible to find exactly the limit of this expression when $\delta t = 0$. If we could get the limit $\left(\text{which is expressed by } \frac{ds}{dt}\right)$, we should find the actual velocity when the train passes a given point at the station. Thus, if s represents the space moved over by a body, and t the time measured from some convenient instant, then the velocity, or the rate of increase of space with time, is denoted by $\frac{ds}{dt}$.

In many cases it is possible to express the relation between s and t by means of a formula, and hence to find the value of $\frac{ds}{dt}$ from the known motion of the body. For example, in the case of a falling body starting from rest at a time when $t=0$, we have

$$s = \tfrac{1}{2}gt^2,$$

where $g = 32.2$ feet per second per second;

$$\frac{ds}{dt} = \mathrm{Lt}_{\delta x=0}\frac{\frac{1}{2}g(t+\delta t)^2 - \frac{1}{2}gt^2}{\delta t}$$
$$= gt.$$

As $\frac{ds}{dt}$ simply denotes velocity, we may replace it by v and thus obtain the well-known law,

$$v = gt \qquad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

In the preceding consideration, v indicated the rate of change of space with time, so, in the same manner, the acceleration of a moving body, which may be denoted by a, is the rate of change of velocity with the time,

$$a = \frac{dv}{dt}.$$

From (i), $$\frac{dv}{dt} = \mathrm{Lt}_{\delta x=0}\frac{g(t+\delta t) - gt}{\delta t}$$
$$= g.$$

Thus, we arrive at a result already well known, that the acceleration of a falling body is g, a constant

Ex. 1. A body falls from rest according to the law $s=16\cdot1\,t^2$, where s is the space passed over in t seconds. Find the actual velocity of the body when t is 1 second

We may, from the given equation, find the space passed over in a fractional part of a second, and, by dividing the space by the time, obtain the average velocity.

Thus, we may take such values of t as 1 and 1·1, 1 and 1·01, and 1 and 1·001, the approximation being closer and closer to the actual value as the interval is diminished. From time 1 to time 1 1 seconds, the space passed over is, from the given equation,

$$16\ 1\{(1\ 1)^2-1^2\}=3\cdot381 \text{ feet,}$$

described in 0·1 second;

$$\text{average velocity during } 0\ 1 \text{ second}=\frac{3\cdot381}{0\ 1}=33\ 81 \text{ feet per second.}$$

The average velocity during the 0·01 second from $t=1$ to $t=1\ 01$ is

$$16\cdot1\{(1\cdot01)^2-1^2\}\div0\ 01=32\ 361 \text{ feet per second}$$

From $t=1$ to $t=1\ 001$, it is

$$16\ 1\{(1\cdot001)^2-1^2\}\div0\ 001=32\ 2161 \text{ ft per sec.}$$

Taking smaller and smaller intervals of time, we find that the average velocity approaches nearer and nearer to the value 32 2, and ultimately we obtain, when t is one second, the actual velocity as 32 2 feet per second

It should be noticed that if t be taken as 0 99 and 1 01, two values separated by an interval of 1 second, then

$$\begin{aligned}\text{average velocity}&=16\ 1\{1\ 01)^2-(0\ 99)^2\}\div\ 02\\&=32\ 2 \text{ ft per sec ,}\end{aligned}$$

and this result follows no matter how much the two intervals may differ from one second, provided their mean is one second.

This will readily be understood when we remember that for such a law of motion the velocity is proportional to the time.

The preceding results are readily obtained by means of Algebra The coordinates of any point on the curve

$$s=16\cdot1t^2 \quad \ldots\ldots\ldots \quad \text{(i)}$$

may be denoted by (s, t), and those of a point near it by $s+\delta s$ and $t+\delta t$

Substituting these values in (i),

$$s+\delta s=16\ 1(t+\delta t)^2=16\cdot1\{t^2+2t\delta t+(\delta t)^2\} \quad \ldots\ldots \quad \text{(ii)}$$

Subtracting (i) from (ii),

$$\delta s = 32{\cdot}2t\delta t + 16{\cdot}1(\delta t)^2$$

Dividing by δt, $\frac{\delta s}{\delta t} = 32{\cdot}2t + 16{\cdot}1\delta t$. .... (iii)

When δt is made zero, then the last term $16{\cdot}1\delta t$ is zero, and (iii) becomes

$$\frac{ds}{dt} = 32{\cdot}2t$$

Hence, the actual value, when t is 1, is 32·2.

Ex. 2. At the end of a time t seconds it is observed that a body has passed over a distance s feet, reckoned from some starting point. If it is known that

$$s = 5t + 0{\cdot}5t^2, \qquad \text{. . ... (i)}$$

what is the velocity at the time t? Plot the curve

Find the average velocity at a time $t = 4{\cdot}1$, 4·01, 4·001. Hence, find the actual velocity at a time $t = 4$

Assuming values 0, 1, 2, for t, values of s can be found. Thus, when t is 2, $s = 5 \times 2 + 0{\cdot}5 \times 4 = 12$.

Other values of s are tabulated.

t	0	1	2	3	4	5	6	7
s	0	5·5	12	19·5	28	37·5	48	59·5

When t is 4·1, $s = (5 \times 4{\cdot}1) + \{0{\cdot}5 \times (4{\cdot}1)^2\}$

$$= 28{\cdot}905;$$

$$\delta t \text{ is } 4{\cdot}1 - 4 = 0{\cdot}1 \quad \text{and} \quad \delta s = 28{\cdot}905 - 28 = 0{\cdot}905.$$

Hence, $$\frac{\delta s}{\delta t} = \frac{0{\cdot}905}{0{\cdot}1} = 9{\cdot}05$$

Similarly, when t is 4·01, $\delta s = 0{\cdot}09005$;

$$\frac{\delta s}{\delta t} = 9{\cdot}005.$$

When t is 4·001, then

$$s = (5 \times 4{\cdot}001) + \{0{\cdot}5 \times (4{\cdot}001)^2\} = 28{\cdot}0090005;$$

$$\delta s = 0{\cdot}0090005 \quad \text{and} \quad \delta t = 0{\cdot}001;$$

$$\therefore \frac{\delta s}{\delta t} = \frac{0{\cdot}0090005}{0{\cdot}001} = 9{\cdot}0005.$$

It is obvious that, as δt is made less and less, the values of $\frac{\delta s}{\delta t}$ are approaching 9; this is confirmed by simple differentiation.

Thus, if $$s=5t+0{\cdot}5t^2,$$

then $$\frac{ds}{dt}=5+t=9, \text{ when } t \text{ is } 4$$

Hence, the actual velocity, when t is 4, is 9 ft per sec

The following construction is an easy verification The value just obtained for v denotes the tangent of the angle made with the axis of x by the line touching the curve at the point P; using the edge of a set-square and a hard, sharp pencil, such a line as in Fig. 116 may be drawn with some approach to accuracy

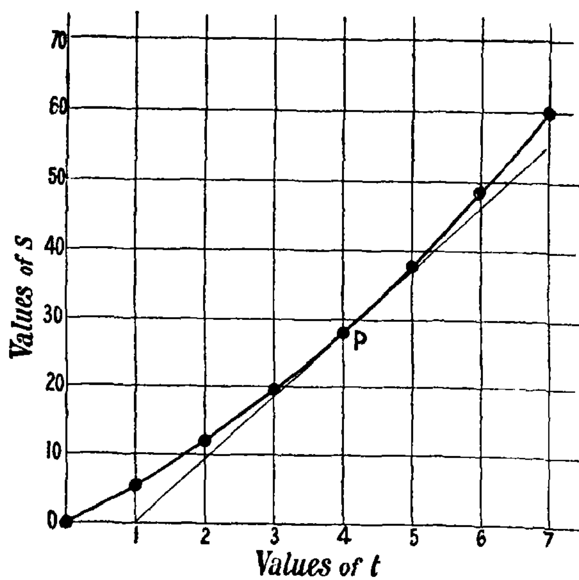

FIG 116 —Slope of a curve

Ex. 3. If $$y=2{\cdot}4-1{\cdot}2x+0{\cdot}2x^2, \qquad (1)$$
find $\frac{dy}{dx}$, and plot two curves from $x=0$ to $x=4$, showing how y and $\frac{dy}{dx}$ depend upon x.

From (i), $$\frac{dy}{dx} = -1{\cdot}2 + 0{\cdot}4x. \quad \ldots \quad \text{(ii)}$$

To plot the two curves given by (i) and (ii), we may, in the usual manner, assume values of x, and calculate values of y.

Thus, from (i), when $x=2$,

$$y = 2{\cdot}4 - 2{\cdot}4 + 4 \times 0{\cdot}2 = 0{\cdot}8.$$

Similarly, when $x=2$, from (ii),

$$\frac{dy}{dx} = -1{\cdot}2 + 2 \times 0{\cdot}4 = -0{\cdot}4.$$

Values of x and y and $\frac{dy}{dx}$ may be tabulated as follows

x	0	1	2	3	4
y	2·4	1·4	0·8	0·6	0·8
$\frac{dy}{dx}$	−1·2	−0·8	−0·4	0·0	0·4

By plotting values of x and y, the curve ab in Fig. 117 is obtained

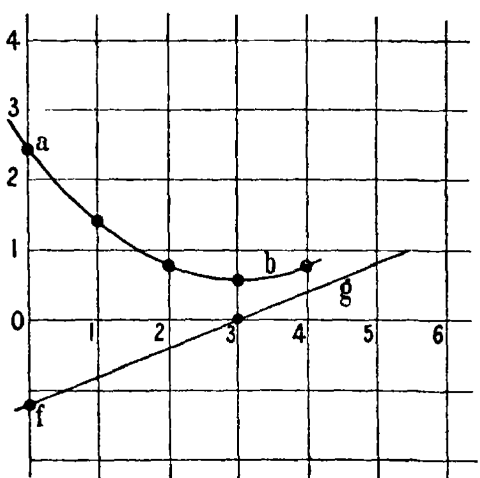

Fig 117.

By plotting the values x and $\frac{dy}{dx}$, the straight line fg (Fig 117) passes through the plotted points.

Force.—In books on Mechanics it is shown that the force F, necessary to give an acceleration a to a body of mass M, is represented by the product of the mass and the acceleration.

$$F = Ma$$

The mass of a body is its weight divided by g, the acceleration of a body falling freely under the action of gravity, where $g = 32{\cdot}2$ ft. per sec per sec

Ex. 4 Find the force required to give a body weighing 100 lbs. an acceleration of 20 ft per sec. per sec.

$$F = \frac{100}{g} \times 20 = 62\ 1 \text{ lbs}$$

[The unit of force is the weight of 1 lb]

Ex 5 A body weighing 100 lbs passes through the space s feet measured from some zero point in its path at the time t seconds, measured from some zero of time; the law of motion is

$$s = 12{\cdot}2 - 3\ 6t + 6{\cdot}7t^2 \qquad \text{(i)}$$

(i) Find the actual velocity at the end of the fourth second

(ii) Find the acceleration and the force which is giving this acceleration to it

Differentiating (i), we obtain

$$v = \frac{ds}{dt} = -3\ 6 + 13\ 4t \qquad \text{..(ii)}$$

Hence, when $t = 4$, $\qquad v = -3\ 6 + 4 \times 13\ 4$

$$= 50 \text{ ft per sec}$$

Let a denote the acceleration, then, from (ii),

$$a = \frac{d^2s}{dt^2} = 13\ 4 \text{ ft per sec per sec}$$

That is, the body increases its velocity at the rate of 13 4 feet every second.

The mass is $100 \div 32\ 2$. If F denotes the force,

then $$F = \frac{100}{32\ 2} \times 13{\cdot}4 = 41{\cdot}61 \text{ lbs}$$

A velocity of 50 ft. per sec is conveniently denoted by 50 *f s* Similarly, an acceleration of 13 4 ft. per sec. per sec would be written 13·4 *f s s*.

In many practical cases the relation between space and time and velocity and time is not known, and an approximate value of $\frac{ds}{dt}$ or $\frac{d^2s}{dt^2}$ is all that can be found. The following example indicates some methods which may be used to find such an approximate value.

Ex. 6. There is a piece of mechanism whose weight is 200 lbs The following values of s in feet show the distance of its centre of gravity (as measured on a skeleton drawing) from some point in its straight path at the time t seconds from some era of reckoning Find its velocity at the time 2·01, its acceleration at the time $t=2·05$ and the force in pounds which is giving this acceleration to it

s	0·3090	0·4931	0·6799	0·8701	1·0643	1·2631
t	2	2·02	2·04	2·06	2·08	2·10

As the values of t differ by 0·02 sec, we may take $\delta t=0·02$, and δs will be obtained by subtracting consecutive values of s This procedure enables values of δs to be tabulated. Thus

$$0·4931-0·3090=0·1841;$$

other values similarly obtained are given in the following table.

Velocity at time 2·01 is 0·1841 ÷ 0·02.

In a similar manner, by subtracting consecutive values of δs, we may obtain the numerical values of $\delta^2 s$. These may be tabulated as follows·

s	0·3090, 0·4931, 0·6799, 0·8701, 1·0643, 1·2631
δs	0·1841, 0·1868, 0·1902, 0·1942, 0·1988.
$\delta^2 s$	0·0027, 0·0034, 0·0040, 0·0046

$$\text{The mean value of } \delta^2 s=\tfrac{1}{4}(0·0027+0·0034+0·0040+0·0046)$$
$$=0·0037.$$

$$\text{Acceleration}=\frac{\delta^2 s}{\delta t^2}=\frac{0·0037}{(0·02)^2}=\frac{0·0037}{0·0004}$$
$$=9·25 \text{ ft. per sec. per sec}$$

As mass is $\frac{200}{32·2}$, force $=\frac{200}{32·2}\times 9·25=57·5$ lbs.

Circular motion.—When a particle of mass m is moving in a circular path of radius r with velocity v, or with an angular velocity ω, in passing from a position P to P_1, although the magnitude of the velocity is unaltered, the direction is changed from that of the tangent at P (Fig 118) to that of the tangent at P_1 The change in the direction of the vector V may be set out, as in Fig 118, by making Oa and Ob each equal in magnitude to v, the former parallel to the tangent at P, the latter to the tangent at P_1, the total *vector change is represented by* the line ab. But it is obvious that ab is made up of a series of vectors obtained by taking points P_2 and P_3, etc., between P and P_1. The result becomes nearer and nearer to the actual value as the points P_2, P_3, etc, approach each other Finally, when P_2, P_3, etc, are consecutive points on the circle, then the vector change at any instant is an indefinitely small arc of a circle of radius v Thus, the vector change, or acceleration, is in the direction of the tangent at a, and is therefore along the radius PC

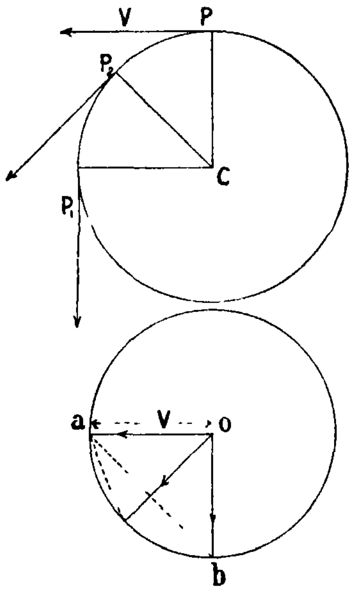

FIG 118 —Motion in a circle

To find the magnitude, let t be the time, in seconds, of one revolution of P. Then, from the relation $s=vt$, we obtain

$$2\pi r = vt\,; \qquad t=\frac{2\pi r}{v} \qquad \text{(i)}$$

Also ...... (vector change per unit time) $\times\, t = 2\pi v$,

or $$\text{acceleration} = a = \frac{2\pi v}{t}.$$

Substitute the value of t from (i);

$$a=\frac{v^2}{r}$$

Harmonic motion.—If a point P (Fig. 119) is moving in a circular path of radius r with uniform speed v ft. per sec., then the acceleration of P at any instant is directed towards C, and its magnitude is given by $\frac{v^2}{r}$.

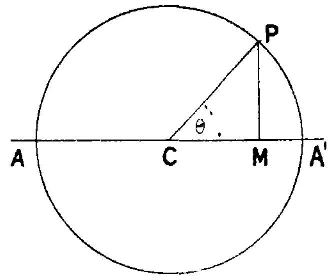

FIG 119.—Harmonic motion

The point M (Fig. 119), the projection of P on a diameter AA', moves with **simple harmonic motion**, usually denoted by the letters S.H.M.

The acceleration of M is the resolved part of the acceleration of P, and is therefore

$$\frac{v^2}{r}\cos\theta = \omega^2 r\cos\theta,$$

where ω denotes the constant angular velocity of P, and θ is the angle PCM.

Let x denote the distance CM, i.e. the distance of M from its mean position.

Then, the acceleration of

$$M = \omega^2 r \times \frac{x}{r} = \omega^2 x \quad \ldots\ldots \text{(1)}$$

If the direction C to A' in the usual manner be taken to be positive, then (1) becomes $-\omega^2 x$, indicating that the direction of the acceleration is from A' to C.

The maximum value of x occurs when P is at A or A', where $x=r$. Hence, maximum acceleration of M is $\omega^2 r$.

Since Force = Mass × Acceleration, it follows from (1) that the force F, acting on a body of mass m moving with S.H.M. is given by $F = m\omega^2 x$.

The maximum value of the velocity occurs when M passes through C.

When a point is moving with S.H.M. the maximum velocity may be obtained by multiplying its mean velocity by $\frac{\pi}{2}$.

If v is the velocity of the point P in the auxiliary circle, the maximum velocity of M occurs when M is at the middle of its path, and is then v or ωr.

If T is the periodic time of a vibration, then

$$\omega=\frac{2\pi}{T},$$

$$\text{mean velocity}=\frac{\text{distance}}{\text{time}}=\frac{4r}{\frac{2\pi}{\omega}}=\frac{2\omega r}{\pi},$$

also
$$\frac{2\omega r}{\pi}\times\frac{\pi}{2}=\omega r=\text{max vel}$$

Ex. 7. A point has two harmonic motions, in the same line, represented by

$$a\sin\frac{\pi t}{2} \text{ and } a\sin\left(\frac{\pi t}{2}+\frac{\pi}{2}\right) \text{ respectively},$$

find the greatest velocity of the resultant motion

Let R denote the resultant velocity;

$$\therefore\ R=a\sin\frac{\pi t}{2}+a\sin\left(\frac{\pi t}{2}+\frac{\pi}{2}\right),$$

or, if
$$\phi=\frac{\pi t}{2},$$

then
$$R=a\sin\phi+a\sin\left(\phi+\frac{\pi}{2}\right).$$

To find the maximum value differentiate and equate to zero in the usual manner (see p 356);

$$\frac{dR}{d\phi}=a\cos\phi+a\cos\left(\phi+\frac{\pi}{2}\right),$$

$$a\cos\phi+a\cos\left(\phi+\frac{\pi}{2}\right)=0,$$

or
$$\cos\phi=-\cos\left(\phi+\frac{\pi}{2}\right)=\sin\phi,$$

$$\tan\phi=1,\ \text{giving}\ \ \phi=45^\circ$$

Hence,
$$R=\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}=a\sqrt{2}$$

We may obtain the same result as follows

$$a\sin\phi+a\sin\left(\phi+\frac{\pi}{2}\right)=a\sqrt{2}\sin\left(\phi+\frac{\pi}{4}\right),$$

$$\therefore \text{ maximum value is } a\sqrt{2}$$

The direction of motion of P is usually taken to be in the opposite direction to the hands of a clock, or anticlockwise;

but in dealing with (say) the mechanism of a direct-acting engine, no such restriction is necessary; the motion may and often does occur in a clockwise direction.

If, as in Fig 120, a rod PQ be attached to P, the direction of motion of Q being always in the line QC, then the motion of Q, for uniform motion of P is not S H M but approaches more to it the longer the link PQ becomes The maximum values of the acceleration of Q occur when P is at A or A', and are given in magnitude by the formula $\omega^2 r\left(1 \pm \frac{r}{l}\right)$.

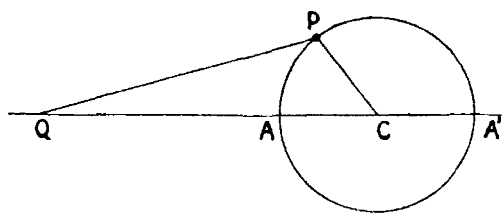

FIG 120

The maximum forces acting on Q therefore occur when P is at A or A', and are, in each case, the product of the mass of the reciprocating parts and the acceleration.

It will be noticed that when l is great compared with r, the term $\frac{r}{l}$ becomes very small and may be neglected; the acceleration may be taken to be simply $\omega^2 r$ Such a case occurs in an eccentric and valve rod in which the motion of the valve is often assumed to be S H M.

The case when the motion of Q is assumed to be S H M. is usually referred to as a rod of infinite length, or more shortly as an infinite rod. When the rod is comparatively short, say 2, 3, 4, etc, times the length of the crank, then the preceding equation may be used to find the magnitude of the maximum acceleration of Q, and hence of the maximum force at Q.

In the formula $m\omega^2 r\left(1 \pm \frac{r}{l}\right)$, where m is the mass of the reciprocating parts, ω the angular velocity of the crank assumed to be constant, l the length of the rod PQ (Fig. 121), and r the length of the crank CP.

Let the crank PC make an angle θ with QC, and let ϕ denote the angle PQC. From P, draw PD perpendicular to QC, and let $PD=y$.

If x denote the distance QC, then

$$x=QD+DC=l\cos\phi+r\cos\theta,$$

$$\frac{dx}{dt}=-l\frac{d\phi}{dt}\sin\phi-r\frac{d\theta}{dt}\sin\theta \quad . \quad . \quad (\text{i})$$

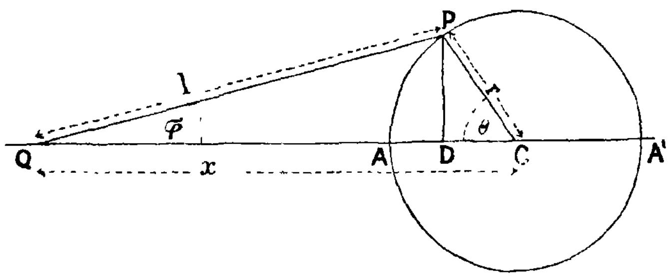

FIG 121

If ω' denote the angular velocity of Q,

then $$\frac{d\phi}{dt}=\omega' \text{ and } \frac{d\theta}{dt}=\omega,$$

by differentiating (i) with regard to t,

$$\frac{d^2x}{dt^2}=-l\frac{d^2\phi}{dt^2}\sin\phi-l\left(\frac{d\phi}{dt}\right)^2\cos\phi-r\frac{d^2\theta}{dt^2}\sin\theta-r\left(\frac{d\theta}{dt}\right)^2\cos\theta \quad (\text{ii})$$

P is a point on the rod PQ and also on the crank CP. Hence, as $v=\omega r$,

$$\omega' l=\omega r\,; \qquad \omega'=\frac{\omega r}{l}$$

When P is at A, $\phi=0$ and $\theta=0$, substitute in (ii);

$$\frac{d^2x}{dt^2}=-\frac{l\omega^2r^2}{l^2}-\omega^2r=-\omega^2r\left(1+\frac{r}{l}\right),$$

and when P is at A', $\phi=0$ and $\theta=\pi$,

$$\therefore \frac{d^2x}{dt^2}=-\omega^2r\left(1-\frac{r}{l}\right)$$

In each of these expressions the negative sign indicates that the direction of the acceleration is negative, i.e tending to decrease x.

Ex. 8. In a direct-acting engine (Fig. 120) the crank CP is 0·5 feet long and makes 125 revolutions per minute. The mass of the reciprocating parts is m. Find the forces acting at Q when the point P is at a dead-point, A or A',

(a) when the connecting rod is infinite,

(b) when the length of the connecting rod is three times the crank

(a) Here $\omega = \dfrac{2\pi \times 125}{60} = \dfrac{125\pi}{30}$ radians per sec.,

$$F = \frac{m \times (125\pi)^2}{30^2} \times 0.5$$
$$= m \times 85.7;$$

(b)
$$F = m\omega^2 r\left(1 + \frac{r}{l}\right)$$
$$= m \times 85.7(1 + \tfrac{1}{3}), \text{ or } m \times 85.7(1 - \tfrac{1}{3})$$
$$= m \times 85.7 \times \tfrac{4}{3}, \text{ or } m \times 85.7 \times \tfrac{2}{3}$$
$$= 114.2m, \text{ or } 57.1m$$

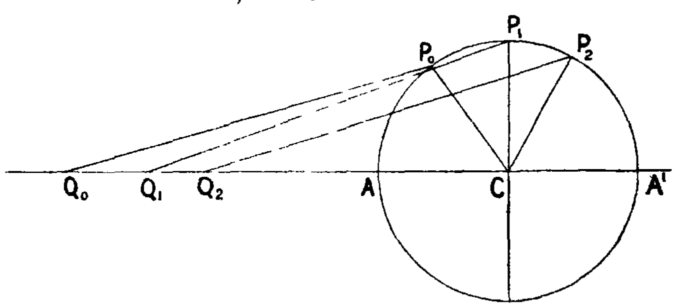

FIG. 122

Graphical methods.—The velocity and acceleration of Q may be obtained by assuming P to move through small distances P_0P_1, P_1P_2, during small intervals of time δt, and then measuring the distances Q_0Q_1, Q_1Q_2, moved through by Q (Fig 121) The distances moved through by Q may be denoted by x; then, subtracting consecutive values, we obtain values of δx Proceeding in this manner, a series of such distances moved through by Q may be obtained and tabulated. From such a table, values of $\dfrac{\delta x}{\delta t}$ can be calculated. Similarly, values of $\dfrac{\delta v}{\delta t}$ or $\dfrac{\delta^2 x}{\delta t^2}$ can be found; from the

latter results an approximate value of the force acting on Q at any given instant—and producing the acceleration of Q—can be obtained

The method adopted may be seen from the following example

Ex 9. In a direct-acting engine mechanism (Fig 120), $CP=6$ in. ($=0\ 5$ ft.), and $PQ=1{\cdot}5$ ft, the crank CP makes 125 revolutions per min in a clockwise direction. The weight of the reciprocating parts at Q is 100 lbs. Find the magnitude of the force at Q for a given position of P.

To obtain the distances moved through by Q draw a diagram (Fig 122) to a scale (say) of 0 1 in. $=1$ ft. The circle denoting the path of the crank pin P may be divided into 24 equal parts, corresponding to equal angular intervals at 15°

To determine the position of Q when P is at point (23) on the circle, use the point as centre and the length of the rod $=1{\cdot}5$ ft. as radius, and describe an arc of a circle; then the point of intersection of the arc with the line CQ gives the position of Q Similarly, using the point 24, or 0, as centre and with the same radius, obtain the next position of Q, and so on. In this manner, the distances moved through by Q, as P moves through equal angular distances of 15°, can be obtained and the distance of each position of Q from some point in QC may be measured and denoted by x.

The time taken by the point P to move through equal angles of 15°, or $\frac{1}{24}$th of the circumference, is $\frac{1}{24}$ (time of one revolution) $=0{\cdot}02$ second.

This may be denoted by δt, and the results tabulated as follows

Position of P No	Displacement of Q $=x$ feet	δx	δt	Velocity $\frac{\delta x}{\delta t}$	δv	Acceleration $a=\frac{\delta v}{\delta t}$
23	0·022					
		−0 022	0 02	−1 10		
0	0·000				2 20	110
		0 022	0 02	1·10		
1	0·022				2·10	105
		0 064	0·02	3 20		
2	0·086				1 75	87 5
		0 099	0 02	4·95		
3	0·185				1 20	60 0
		0·123	0·02	6·15		
4	0 038					

By taking the differences of the various tabulated values of x in column 2, a series of values δx, as in column 3, are obtained The ratio $\frac{\delta x}{\delta t}$ gives approximately the velocity of Q at each given instant

In like manner, by taking the differences of consecutive values of v, column 6, giving numerical values of δv, can be obtained Finally, the acceleration at each position is approximately given by $\frac{\delta v}{\delta t}$

If W denotes the weight of the reciprocating parts, then $W \div g$ is the mass, and when W is known, the force acting at any point of the stroke can be ascertained

EXERCISES XXXVIII.

1 A body is observed at the instant when it is passing a point P From subsequent observations it is found that in any time t seconds, measured from this instant, the body has described s feet (measured from P) where s and t are connected by the equation $s = 2t + 4t^2$. Find the average speed of the body between the interval $t = 1$ and $t = 1.1$ between $t = 1$ and $t = 1.001$ and between $t = 1$ and $t = 1.0001$ and deduce the actual speed when t is exactly 1.

2. Suppose that a curve has been plotted such that the ordinates and abscissae represent distance and time respectively, what will be represented by the slope of the curve at any point on it? Obtain an expression for the slope if the distance s and time t are connected by the equation
$$s = 5t + 2.1t^2.$$
Give the numerical value at the instant when $t = 5$

3 At the end of a time t seconds it is observed that a body has passed over a distance s feet reckoned from some starting point. If $s = 25 + 150t - 5t^2$, find the velocity at a time t and give the value when $t = 7$ Find also the acceleration and the force causing this acceleration if the weight of the body is 100 lbs

4 A train starts from rest and its speeds at the ends of the first, second, third, fourth, fifth and sixth minutes are 9.8, 13.75, 16.95, 19.6, 21.9 and 24 miles per hour respectively Plot a curve showing the relation between speed and time, and between acceleration and time, deduce approximately the velocity and acceleration at the end of the sixth minute

5. A body has passed through the space s feet measured from some zero point in its path at the time t seconds measured from some zero of time; the law of motion is
$$s = 12.2 - 3.6t + 6.7t^2.$$

Calculate the average velocity of the body

(i) for the next tenth of a second following the completion of the fourth second.

(ii) for the next $\frac{1}{100}$th of a second following the completion of the fourth second.

(iii) for the next $\frac{1}{1000}$th of a second following the completion of the fourth second

Hence deduce the actual velocity at the end of the fourth second

6. A piston makes n revolutions per second and drives a crank of length r through a connecting rod of length l Show that the acceleration at the ends of the strokes are

$$4\pi^2n^2r\left(1+\frac{r}{l}\right) \text{ and } 4\pi^2n^2r\left(1-\frac{r}{l}\right).$$

7. A body weighing 50 lbs has passed through the space s feet measured from some zero point in its path at the time t seconds measured from some zero of time, the law of motion is

$$s=1\cdot 2-0\cdot 6t+1\cdot 7t^2$$

Find the acceleration when t is 7 and the force giving this acceleration to it.

8. The following values of s, in feet, show the distance of the centre of gravity of a piece of mechanism weighing 100 lbs from some point in its straight path at the time t seconds Find the velocity and the acceleration at the time $t=0\cdot 085$, find also the force which is giving this acceleration to it.

s	0 088	0 2226	0 3612	0 5038	0 6505	0 8011
t	0 06	0 07	0 08	0 09	0 10	0 11

9 In the mechanism shown (Fig 123) C and D are fixed centres of motion, the linear scale of the figure being $\frac{1}{8}$ full size, CB is a crank (6″ long) rotating in a clockwise direction at a speed of 8 radians per sec DA is an oscillating lever and AB a connecting link Draw a diagram which shall give the acceleration of any point in the link BA, and state the magnitude and direction of the acceleration of the point E.

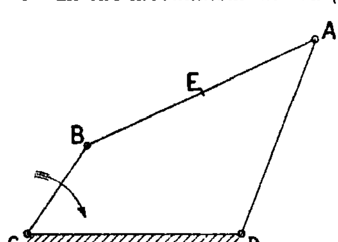

FIG. 123

10. In a direct-acting engine mechanism (Fig 120) a crank CP rotates about a fixed centre C, and the end of the connecting rod PQ moves in the line QC.

Given $CP=5$ in., $PQ=16$ in.; speed 120 revolutions per min.

Find by means of careful graphical construction, measurement, tabulation, and calculation, the displacement, velocity and acceleration of Q as P moves through equal distances of $\frac{1}{24}$th the circumference.

Complete the following table :

Position of P	Displacement of $Q=x$ feet	δx	δt	Velocity $v=\delta x \div \delta t$	δv	Acceleration $a=\delta v \div \delta t$
0	0					
			0·0208	0·87		
1	0·0183				1·67	80·3
		0·0183				
2	0·0725		0·0208	2·54		
		0·0542				
3	0·1542					
4						

11. The sketch (Fig. 124) shows a mechanism called a "quick return motion," where CP is a crank rotating with constant speed, the end of the rod PQ moving in the straight line QM.

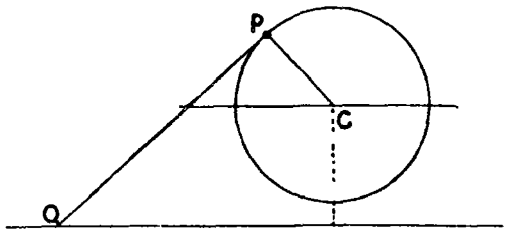

FIG 124

Given $CP=5$ in., $PQ=16$ in., and $CM=7$ in.; speed 120 revolutions per min; determine as in the preceding exercise the displacement, velocity, and acceleration of the point Q.

Set out curves representing these quantities (*a*) on a time base, (*b*) on a displacement base.

12. CP (Fig 121) is a crank which rotates clockwise about C at a uniform speed of 1·5 radians per second. PD is a perpendicular on a fixed horizontal line. The position shown is that for which the time $t=0$; the figure is $\frac{1}{4}$ full size.

If y is the distance of D from C at any time t (positive when to the right of C) and is given by $y=a \sin(qt+e)$, find the numerical values of a, q and e in this case. Also draw the position of the crank and of D when $t=3$, and measure the value of y.

13 In Fig. 125, a diagram of a radial valve gear is given, the point Q moving in the straight line XQ.

Given $CP=5''$, $PR=QM=12''$, $RQ=14''$, $RS=4''$, angle $PSL=30°$.

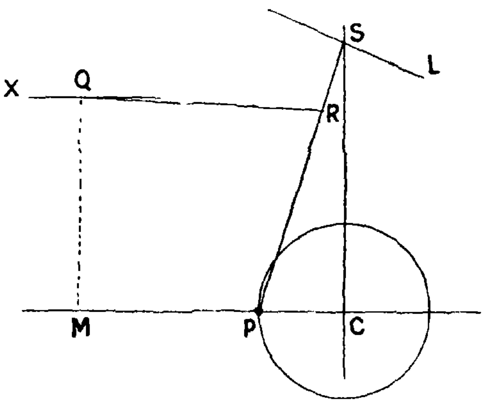

FIG 125

Find the displacement, velocity, and acceleration of Q for a number of consecutive positions of P when the speed of the crank CP is 120 revolutions per min.

14. The successive positions of a piston at intervals of time of $\frac{1}{40}$th of a second are 0·0, 0·024, 0·097, 0·206, 0·341 feet respectively

Determine (i) the velocity and (ii) the acceleration of the piston at successive intervals Draw diagrams showing the velocity and acceleration at any time. Read off the acceleration when $t=0·05$ seconds

CHAPTER XVII.

MAXIMA AND MINIMA

Maxima and minima—It has already been shown (p 306) that the slope of the curve representing $y=f(x)$ is equal to $\frac{dy}{dx}$ In Fig. 126 the graph of a function $y=f(x)$ is shown, and the changes in the slope of this curve may be seen from the varying inclinations of the lines touching the curve at various points

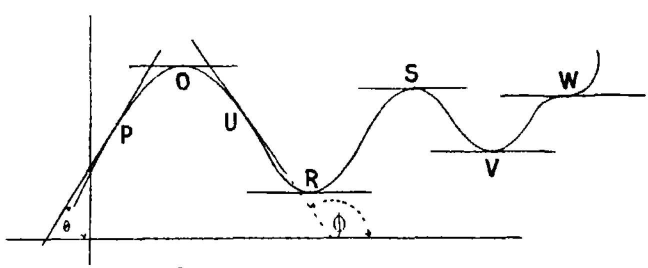

FIG 126—Maxima and minima.

Thus, at a point P, $\frac{dy}{dx}=\tan\theta$, and as θ is less than 90° the slope of the curve at P is positive, *i.e.* $\frac{dy}{dx}$ is positive At U, $\frac{dy}{dx}=\tan\phi$ and is negative. If the curve has been continuous between P and U, then $\frac{dy}{dx}$ must have had a zero value at some intermediate point, or in other words, the tangent to the curve must have been parallel to the axis of x Such a point is shown at Q At each of the points R, S, V, and W, $\frac{dy}{dx}$ must

also be zero It will be seen that the ordinate at Q is a little greater than any ordinate near to it on either side; it is said to be a *maximum* ordinate, or a maximum value of y.

The ordinate at V is less than any adjacent to it on either side, and is called a *minimum* ordinate.

DEF. When y increases with increase of x to a certain value and then diminishes, it is said to have a **maximum** value where the change occurs; and when y diminishes to a certain value and then increases, a **minimum** value is obtained In either case $\frac{dy}{dx}=0$ So the maximum value of a function may be defined as a value greater than either the one just before it or just after it Or, in other words, $\frac{dy}{dx}$ changes from + to − as the curve passes through a maximum point Similarly, if $\frac{dy}{dx}$ changes from − to + in passing through zero, the point where $\frac{dy}{dx}$ *is zero is a minimum point*

Points of inflection.—It should be noted that although $\frac{dy}{dx}$ must be zero whenever y is a maximum or minimum, it does not follow that if $\frac{dy}{dx}=0$ that y must have a maximum or minimum value at that point Thus, at W, Fig 126, $\frac{dy}{dx}=0$, because the tangent there is parallel to the axis of x, yet y is neither a maximum, nor a minimum At such a point, called **a point of inflection**, it will be found that $\frac{dy}{dx}$ does not change sign in passing through zero

It will be seen from Fig 126 that the terms maximum and minimum are relative, and that we can have one maximum value, as at Q, greater than another maximum, as at S

The method of procedure in finding maximum or minimum values of a function y will be seen in the following example:

Ex. 1. Find for what values of x the function

$$y=x^3-6x^2+9x-12,$$

is a maximum or a minimum. Give the maximum and minimum values of y.

Since $$y=x^3-6x^2+9x-12; \quad \ldots \quad (1)$$

$$\therefore \frac{dy}{dx} = 3x^2 - 12x + 9.$$

But when y is a maximum or minimum, $\dfrac{dy}{dx} = 0$.

To find what values of x make $\dfrac{dy}{dx}$ zero, we solve the equation

$$3x^2 - 12x + 9 = 0,$$

and obtain $\qquad x = 1$ and $x = 3$.

It remains to determine which of these values makes y a maximum and which makes it a minimum

In Eq. (i), substitute $\qquad x = 1$;

$$y = 1 - 6 + 9 - 12 = -8\ ; \quad \therefore y = -8.$$

Now, when $x = 0{\cdot}999$, a value slightly less than 1, find the value of y;

$\therefore y = -8{\cdot}000003$

Also, when

$x = 1{\cdot}001$,

$y = -8{\cdot}000003$

Hence y increases, algebraically, as x increases from $0{\cdot}999$ to $x = 1$, and diminishes as x increases from 1 to $1{\cdot}001$ (since $-8{\cdot}000003$ is < -8).

Hence, at $x = 1$, y has the maximum value -8

Another method of testing will be applied at $x = 3$.

In Fig 127 it is evident that $\dfrac{dy}{dx}$ is positive for a value of x slightly less than that giving y a maximum, and negative for a value of x a little greater than this; also $\dfrac{dy}{dx}$ is negative for x less than, and positive for x greater than, that making y a minimum.

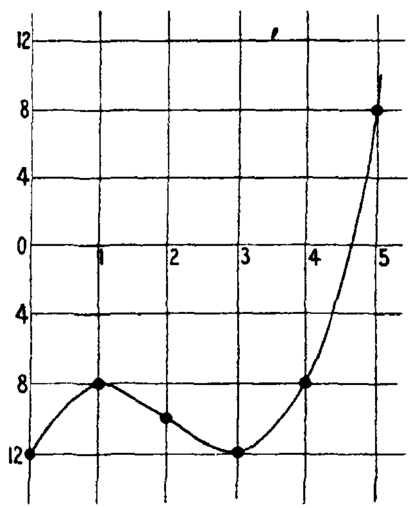

FIG 127 Graph of $y = x^3 - 6x^2 + 9x - 12$

Now, $\qquad \dfrac{dy}{dx} = 3(x^2 - 4x + 3)$;

when $\qquad x = 2{\cdot}99, \ \dfrac{dy}{dx} = -0{\cdot}0603$;

when $x=3{\cdot}01$, $\dfrac{dy}{dx}=+0{\cdot}0603$,

or $\dfrac{dy}{dx}$ changes from $-^{ve}$ to $+^{ve}$ as x increases from 2·99 to 3·01.

Hence $x=3$ gives a minimum value of $y=-12$. Fig. 127 shows the graph of $y=x^3-6x^2+9x-12$.

It will be noticed in Fig 126 that maximum and minimum values of y occur alternately. This is always so; between two consecutive maximum values of y there must be one, and only one minimum value, and between consecutive minimum values, one maximum. For after y is a maximum it decreases and must, before it can increase again to reach another maximum, have stopped decreasing, and so have had a minimum value

By plotting a function we can always find maximum and minimum values and this is often the readiest and simplest method available; in the case of experimental numbers it is the only method.

Ex. 2. $y=\dfrac{(x-1)^3}{(x+1)^2}$ Find maximum and minimum values of y

We find $$\frac{dy}{dx}=\frac{(x-1)^2}{(x+1)^3}(x+5).$$

Hence, $x=1$ and $x=-5$ both make $\dfrac{dy}{dx}$ zero

When $x=1-h$, $y=-\dfrac{h^3}{(2-h)^2}$,

and when $x=1+h$, $y=\dfrac{h^3}{(2+h)^2}$;

y increases continuously as x changes from $1-h$ to $1+h$, so $x=1$ cannot make y either a maximum or minimum Apply the same test at $x=-5$, we find that y is a maximum there.

Ex. 3. If $$y=\sin^3\theta\cos\theta, \quad \ldots \quad \text{(i)}$$
show that y is a maximum when $\theta=60°$.

Substituting various values, 10°, 20°, etc, for θ, the corresponding values of y can be calculated from (i).

Thus, when $\theta=40°$,

$$\sin 40°=0{\cdot}6428, \quad \cos 40°=0{\cdot}7660,$$
$$y=(0{\cdot}6428)^3\times 0{\cdot}7660=0{\cdot}2033.$$

Other values of y may be obtained in like manner and tabulated as follows ·

θ	20°	40°	60°	80°	90°
y	0 0376	0 203	0 325	0 166	0

Plotting these values as in Fig 128, the maximum value of y occurs at m when $\theta = 60°$.

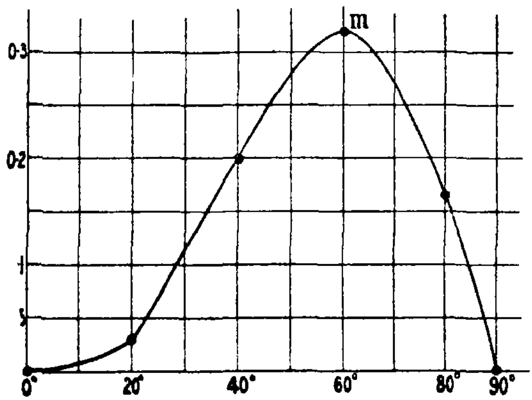

FIG 128 —Graph of $y = \sin^3\theta \cos\theta$

We have $$y = \sin^3\theta \cos\theta\,;$$

$$\frac{dy}{dx} = -\sin^3\theta \sin\theta + 3\cos\theta \sin^2\theta \cos\theta = -\sin^4\theta + 3\sin^2\theta \cos^2\theta,$$

for a maximum value this must vanish ;

$$3\sin^2\theta \cos^2\theta - \sin^4\theta = 0$$

The solutions of this equation are $\theta = n\pi$ or $n\pi + (-1)^{n-1}\frac{\pi}{3}$.

This gives $\theta = 60°$.

Ex. 4. To divide a given number into two parts so that their product is a maximum.

Let a be the given number, and x one of the parts, then the remaining part is $a - x$. The product is $x(a - x)$

If $$y = x(a - x) = ax - x^2,$$

for a maximum.

By differentiation, $\frac{dy}{dx}=a-2x=0$ for a maximum;

$$x=\frac{a}{2}$$

A result which gives a maximum value of y, as may easily be proved.

Hence, the two parts must be equal

It will be noticed that this is the same problem as to divide a line into two parts such that the rectangle on the two parts as sides is a maximum Hence, of all rectangles having a given perimeter, the square has the greatest area

Application to a beam.—The strength of a rectangular beam to resist cross-breaking is known to vary as bd^2, where b is the breadth, and d the depth

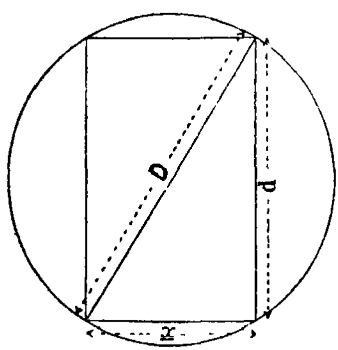

FIG 129

The value of x, the breadth of a beam of a maximum strength which can be cut from a circular log of diameter D (Fig. 129), may be obtained either by plotting or by differentiation

Thus, if d be the depth, then $d=\sqrt{D^2-x^2}$, and putting

$$y=bd^2=x(D^2-x^2), \qquad (i)$$

we obtain $$\frac{dy}{dx}=D^2-3x^2,$$

and therefore for a maximum $\left(\text{i e } \frac{dy}{dx}=0\right)$,

$$x=\frac{D}{\sqrt{3}} \qquad \ldots \quad (ii)$$

Ex. 5. Let the diameter D be 9 in Then, giving a series of values to x, values of y can be calculated and tabulated as follows:

x	0	1	2	3	4	5	6	7	8	9
y	0	80	154	216	260	280	270	224	136	0

By plotting the values of x and y a curve may be drawn through the plotted points as in Fig 130. The maximum value, *i.e.* the point on the curve at which the tangent is horizontal, is seen to be between $x=5$ and $x=6$, viz. at a. Also, from such a curve, we can find within what limits the breadth may vary so as not to weaken the beam more than a certain percentage, say 10 or 15 per cent

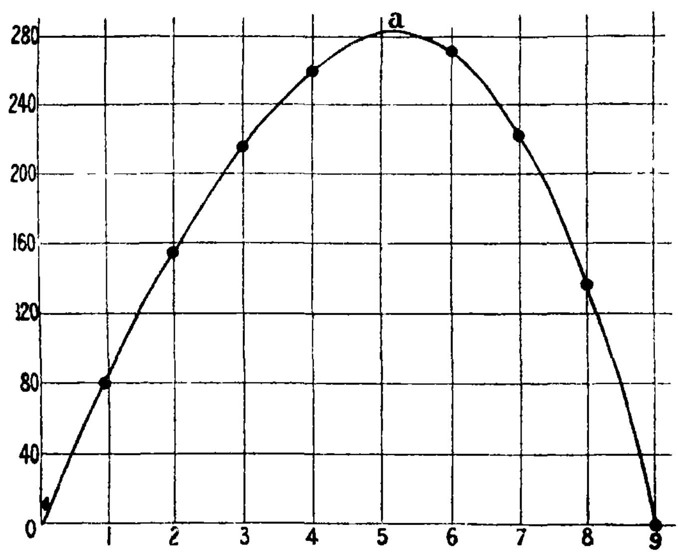

Fig. 130 —Graph of $y=x(81-x^2)$

Now, making $D=9$ in (ii), we have

$$x=\frac{9}{\sqrt{3}}=3\sqrt{3}=5\ 196 \text{ in.}$$

The maximum value of y can readily be obtained either from the curve or by substituting the value of x in (i)

Thus, $\qquad y=3\sqrt{3}(81-27)=162\sqrt{3}=280\ 58$

Stiffest beam.—The deflection of a beam due to a given load is proportional to the breadth and the cube of the depth of the beam

Ex 6. If D is the diameter of a cylindrical log of timber, and if x denote the breadth, then the depth d is $\sqrt{D^2 - x^2}$

Hence, putting $y = xd^3$;

$$y = x(D^2 - x^2)^{\frac{3}{2}},$$

$$\frac{dy}{dx} = (D^2 - x^2)^{\frac{3}{2}} + \frac{3}{2}x(D^2 - x^2)^{\frac{1}{2}} \times (-2x)$$

$$= (D^2 - x^2)^{\frac{1}{2}}\{D^2 - 4x^2\}$$

For the stiffest beam $\frac{dy}{dx}$ must vanish, giving $x = \frac{D}{2}$, the remaining value $x = D$ being obviously inadmissible

Ex 7. The two banks of a lake are parallel and 100 yds apart A person at a point A (Fig 131) on one bank wishes to reach a point B 300 yds. ahead of him on the opposite bank in the shortest possible time. If he can travel on the bank AC at the rate of 5 miles an hour and can row at 3 miles an hour, at what point D in AC should he begin to row?

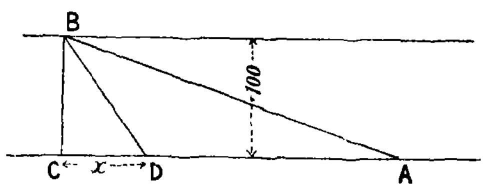

FIG 131

Draw CB perpendicular to AC and let the distance CD be denoted by x Then, $AD = 300 - x$

The distance $DB = \sqrt{100^2 + x^2}$.

and time taken from D to B is

$$\frac{\sqrt{100^2 + x^2}}{3}.$$

Along the bank the distance $AD = 300 - x$ and time taken from A to D

$$= \frac{300 - x}{5}.$$

The total time $$t=\frac{\sqrt{100^2+x^2}}{3}+\frac{300-x}{5}$$

$$=\frac{5\sqrt{100^2+x^2}+900-3x}{15}$$

is to be a minimum;

$$\frac{dt}{dx}=\frac{5\times\frac{1}{2}(100^2+x^2)^{-\frac{1}{2}}\times 2x-3}{15}=0$$

for a maximum or minimum,

whence $$\frac{5x}{\sqrt{100^2+x^2}}-3=0$$

Hence, $16x^2=9\times 100^2, \qquad \pm x=75$ yds.

It is obvious that the negative value is not applicable, hence $x=75$ yds.

Ex. 8 **Height of rectangle of maximum area inscribed in a given triangle.**

Let ABC (Fig. 132) be the given triangle, the base AB equal to a, and the altitude h.

Let GD, one of the sides of the rectangle, be denoted by x, and the base, FG, by y.

Height of triangle $=h-x$,

and $h : (h-x)=AB : DE$ (similar $\triangle$s),

or $h : h-x=a : y$;

$$\therefore \quad y=\frac{a(h-x)}{h}.$$

Area of rectangle

$$=x\times y=\frac{a}{h}(h-x)x;$$

$$A=ax-\frac{ax^2}{h},$$

$$\frac{dA}{dx}=a-\frac{2ax}{h}=0$$

for a maximum or minimum,

giving $2x=h; \quad \therefore \quad x=\frac{h}{2}$,

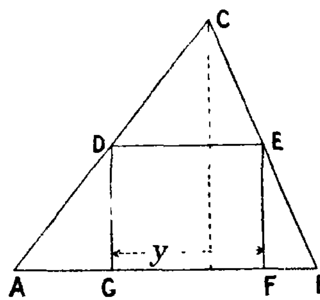

Fig. 132.—Rectangle of maximum area inscribed in a triangle.

which makes A a maximum; therefore altitude of rectangle must be one-half of the altitude of the triangle

Ex. 9. **To find the dimensions of the cylinder of greatest volume which can be obtained from a given right cone.**

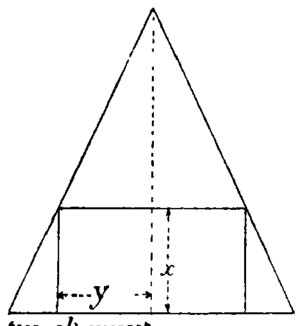

FIG 133 —Cylinder of greatest volume in a cone

Let h denote the height of the cone, and b the radius of the base (Fig. 133) Also let x and y denote the corresponding dimensions for the cylinder

Then $\qquad V=\pi y^2 \times x, \qquad$.(i)

also $\quad h . (h-x)=b \quad y;$

$$y=\frac{b(h-x)}{h}. \qquad \text{(ii)}$$

and $$V=\frac{\pi b^2}{h^2}(h-x)^2 x, \qquad \text{(iii)}$$

as b and h are both constant To obtain the maximum value of (ii) it is only necessary when differentiating to consider the terms $(h-x)^2x$.

Let $\qquad V'=(h-x)^2x=h^2x-2hx^2+x^3.$

Then $$\frac{dV'}{dx}=h^2-4hx+3x^2;$$

$$3x^2-4hx=-h^2$$

Solving this, we find $x=h$ or $\frac{h}{3}$. The former is inadmissible.

Hence, substituting the value $x=\frac{h}{3}$ in (ii),

$$y=\frac{b}{h}\left(h-\frac{h}{3}\right)=\frac{2}{3}b;$$

$$V=\frac{4\pi hb^2}{27}$$

Ex. 10. Show that the expense of lining a cylinder of given volume with lead will be least when the depth of the cylinder is equal to the radius of the base

Let x denote the height and y the radius of the base

The surface S will be the convex surface $2\pi xy$ together with the area of the base πy^2;

$$S=2\pi xy+\pi y^2, \qquad \text{(i)}$$

$$V=\text{volume}=\pi y^2 x;$$

$$\therefore x=\frac{V}{\pi y^2} \qquad \text{(ii)}$$

Substitute in (i); $$S=\frac{2\pi V}{\pi y^2}y+\pi y^2=\frac{2V}{y}+\pi y^2,$$

$$\frac{dS}{dy}=-\frac{2V}{y^2}+2\pi y;$$

$$\therefore y^3=\frac{V}{\pi} \text{ for a minimum.}$$

From (ii), $$x^3=\frac{V^3}{\pi^3 y^6}=\frac{V^3}{\pi^3\times\frac{V^2}{\pi^2}}=\frac{V}{\pi}=xy^2 \text{ from (ii).}$$

Hence $x=y$, or the height of the cylinder is equal to the radius of the base.

We may consider the preceding problem as an example of a more general method Thus, taking the equations for the surface and volume respectively of a cylinder,

$$S=\pi y(2x+y),$$

where x denotes the slant height of cone and y the radius of its base,

$$V=\pi y^2 x$$

Two conditions are to be satisfied. $\frac{dS}{dx}$ must be zero for a minimum (Either x or y might have been chosen as the independent variable.)

Also V is to be constant, or $\frac{dV}{dx}$ must be zero.

$$\frac{dS}{dx}=0 \text{ gives } y\left(2+\frac{dy}{dx}\right)+(2x+y)\frac{dy}{dx}=0, \qquad \text{(i)}$$

and $$\frac{dV}{dx}=0 \text{ gives } y^2+2xy\frac{dy}{dx}=0. \qquad \text{(ii)}$$

To find the relation between x and y eliminate $\frac{dy}{dx}$,

$$\text{from (ii), } \frac{dy}{dx}=-\frac{y}{2x}$$

Substitute this value in (i);

$$\therefore y\left(2-\frac{y}{2x}\right)-(2x+y)\frac{y}{2x}=0,$$

or $$2-\frac{y}{2x}=1+\frac{y}{2x},$$

or $$\frac{y}{x}=1,$$

i.e. $$y=x.$$

Ex. 11. From a circular disc of thin sheet copper a piece in the shape of a sector is cut out in such a way that the remainder can be bent into the form of a right circular conical funnel. What is the least possible diameter for the disc if the capacity of the funnel is to be one pint? [1 pint = 34 66 cub in]

Let r denote the length of a slant side of cone, and x the radius of the base of the cone

$$V = \text{volume of cone} = \frac{\pi}{3}x^2\sqrt{r^2 - x^2}, \qquad \text{(i)}$$

and the volume has to be constant, viz. 1 pint; $\quad \frac{dV}{dx} = 0$

But the minimum value of the diameter, or the radius, being required,

$$\frac{dr}{dx} = 0 \qquad \text{(ii)}$$

From
$$\frac{dV}{dx} = 0$$

we obtain
$$2x\sqrt{r^2 - x^2} + \frac{1}{2}\frac{2r\frac{dr}{dx} - 2x}{\sqrt{r^2 - x^2}} \times x^2 = 0 \qquad \text{(iii)}$$

Substituting from (ii),

$$2(r^2 - x^2) - x^2 = 0,$$

or
$$r = \sqrt{\frac{3}{2}}x \quad \ldots \qquad \text{(iv)}$$

From (i) and (iv), since 1 pint = 34·66 cub in ,

$$34\ 66 = \frac{\pi}{3} \times \frac{2}{3}r^3\left(1 - \frac{2}{3}\right)^{\frac{1}{2}},$$

or
$$r^3 = \frac{9\sqrt{3} \times 34{\cdot}66}{2\pi},$$

$$r = 4\ 413 \text{ in.},$$

or the least diameter is 8 826 inches.

Ex. 12. It is known that the weight of coal in tons consumed per hour in a certain vessel is $0{\cdot}3 + 0\ 001v^3$, where v is the speed in knots (or nautical miles per hour). For a voyage of 1,000 nautical miles, tabulate the time in hours, and the total coal consumption for various values of v If the wages, interest on cost of vessel, etc., are represented by the value of 1 ton of coal per hour, tabulate for each value of v the total cost, stating it in the

value of tons of coal, and plot on squared paper. About what value of v gives the greatest economy?

Let t denote the time (in hours), and s the distance described, then

$$s = vt. \quad \ldots \quad (\text{i})$$

Total cost in tons of coal consumption

$$= C = t + (0.3 + 0.001v^3)t \quad (\text{ii})$$

Also t may be expressed in terms of v from (i);

$$t = sv^{-1} = 1000v^{-1}$$

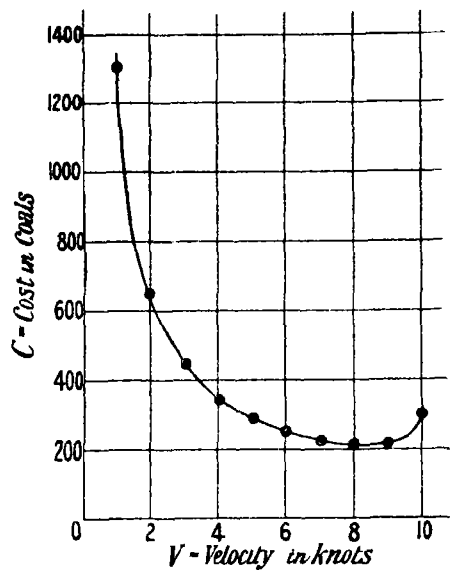

FIG. 134

Substituting in (ii),

$$C = 1000v^{-1} + (0.3 + 0.001v^3)1000v^{-1}$$

$$= 1300v^{-1} + v^2,$$

$$\frac{dC}{dv} = -1300v^{-2} + 2v = 0 \text{ for a minimum;}$$

$$v^3 = 650 \text{ gives a minimum,}$$

hence $\quad v = 8{\cdot}66$ knots.

or, the minimum value for v may be obtained by plotting on squared paper

Tabulating each value of v, we obtain the following table:

v	1	2	3	4	5	6	7	8	9	10
t	1000	500	333	250	200	167	143	125	111	100
Tons of coal consumed	301	154	109	91	85	86	92	101.5	114	130
Total cost -	1301	654	442	341	285	253	235	226.5	225	230

Plotting these values as in Fig 134, it is seen that the total cost C passes through a minimum at a point where $v=8.7$ (roughly)

Ex. 13. Given the perimeter of an ellipse, find the relation between the major and minor axes, so that the area may be a maximum

Denote the axes by x and y. The perimeter of an ellipse cannot be accurately expressed in a simple form, but a rough form usually employed is expressed by $\pi(x+y)$, where x and y denote the semi-major and semi-minor axes respectively

The area of the ellipse $A=\pi xy$ . . . .(i)

If p denote the given perimeter, then

$$p=\pi(x+y)\,; \qquad \text{(ii)}$$

$$\therefore\ y=\frac{p}{\pi}-x$$

Substituting this value in (i);

$$A=\pi x\left(\frac{p}{\pi}-x\right)=px-\pi x^2\,;$$

$$\frac{dA}{dx}=p-2\pi x=0\,;$$

$$x=\frac{p}{2\pi},$$

or, the given ellipse must have its semi-axes equal, and that form is a circle

Prof. Boys, F R S, has suggested the use of elliptical water pipes to prevent the pipes bursting during frosty weather. The expansion of the water due to freezing tends to make the internal cross-section become more circular, that is, to increase its area; and the internal volume of the pipe would be correspondingly enlarged

Ex. 14. When is $x^{\frac{2}{\gamma}} - x^{1+\frac{1}{\gamma}}$ a maximum, γ being 1·4? Also show, by two or three values near the maximum, and on either side, that the value obtained is a maximum

Let $y = x^{\frac{2}{\gamma}} - x^{1+\frac{1}{\gamma}}$

Substituting the given value for γ,

$$y = x^{\frac{2}{1\cdot 4}} - x^{\frac{1+1\cdot 4}{1\cdot 4}};$$

$$y = x^{\frac{10}{7}} - x^{\frac{12}{7}}, \quad \ldots \quad \text{(i)}$$

$$\frac{dy}{dx} = \frac{10}{7}x^{\frac{3}{7}} - \frac{12}{7}x^{\frac{5}{7}}$$

For a maximum value $\frac{dy}{dx} = 0$;

$$\frac{10}{7}x^{\frac{3}{7}} - \frac{12}{7}x^{\frac{5}{7}} = 0,$$

or

$$5x^{\frac{3}{7}} = 6x^{\frac{5}{7}}, \qquad x^3 = (1\cdot 2)^7 x^5;$$

$$x^2 = \frac{1}{(1\cdot 2)^7} = \left(\frac{10}{12}\right)^7,$$

$$2 \log x = 7 (\log 10 - \log 12);$$

$$x = 0\cdot 5282$$

Insert this value in (i) and calculate the corresponding value of y Thus, when $x = 0\cdot 5282$, from (i),

$$y = 0\cdot 5282^{\frac{10}{7}} - 0\cdot 5282^{\frac{12}{7}} = 0\cdot 0669.$$

Calculate values of y on each side of the maximum, and tabulate as follows

Values of x	0·4	0·52	0·5282	0·53	0·6
y	0·0627	0·0668	0·0669	0·0668	0·0652

Ex 15. Determine the speed most economical in fuel when steaming *against* a tide, supposing the resistance to the ship to vary as the square of the velocity, and that the fuel burnt per hour is proportional to the product of resistance and speed.

Let v miles per hour be the constant velocity of the tide, and V miles per hour the velocity of the ship. Then, the velocity of the ship relative to the bank is $V - v$ miles per hour;

$\therefore$ time required to steam a distance of m miles is $\frac{m}{V-v}$ hours

But the resistance to motion is proportional to V^2, and the fuel burnt per hour to $V \times V^2 = V^3$;

$\therefore$ fuel burnt per hour $= KV^3$, where K is some constant;

$\therefore$ to steam m miles requires $F = \frac{m}{V - v} \times KV^3$ lbs. of fuel.

We have to find what value of V makes F a minimum.

$$\frac{dF}{dV} = mK \times \frac{3V^2(V - v) - V^3}{(V - v)^2}$$

$$= mK \times \frac{2V^3 - 3V^2 v}{(V - v)^2}.$$

This is zero when $V = 0$ or $V = \frac{3v}{2}$.

$V = 0$ is inadmissible Hence, $V = \frac{3}{2}v$ gives the speed at which the minimum quantity is burnt.

Taking $v = 5$ miles per hour, and that $K = 0{\cdot}0016$, plot the curve connecting fuel per mile per hour in tons, if $K = 0{\cdot}0016$, and V varies from $V = 5\frac{1}{2}$ to $V = 10$. And show that we can depart considerably from $V = 7\frac{1}{2}$, the most economical speed without altering F very much.

[This is shown by the graph of F and V being flat in the neighbourhood of $V = 7\frac{1}{2}$.]

EXERCISES XXXIX

1 The sum of two numbers is 33; find the numbers when the sum of their squares is a maximum

2 For what value of x is $(3a - 4x^3)$ a minimum? Is there a maximum?

3. Find the turning values of $x + \frac{1}{x}$

4. Find the area of the greatest rectangle whose perimeter is 10 feet.

5. Divide a line into two parts so that the sum of the squares on the two parts shall be a minimum

Find the maximum and minimum values of the following:

6. $3x^4 + 8x^3 - 24x^2 - 96x + 112$

7. $2x^3 - 17x^2 + 44x - 30$.

8 $\frac{x^3}{3x^2 - a^2}$.

9. Prove that the greatest value of

$$\frac{2x\sqrt{9+3x^2}}{9+7x^2} \text{ is } \tfrac{1}{2}.$$

10. Divide 12 into two parts, (i) so that the least multiplied by the square of the greatest shall be a maximum; (ii) so that the least multiplied by the cube of the greatest shall be a maximum.

11. Find maximum and minimum values of $y=\cos(ax+b)$.

12. Find the value of x for which $y=\frac{a}{x}+bx$ is a minimum; find the numerical value when $a=8$, $b=2$

13. Find the maximum and minimum values of

(i) $y=(x-3)^3(x^2-3x-3)$, (ii) $y=x^3(x-4)$, (iii) $y=x^{2n+1}(x-2n)$.

14 Find maximum and minimum values of $\sqrt{a+x}+\sqrt{a-x}$.

15 Find the least area of sheet metal that can be used to make a cylindrical gasometer, whose volume is 10 million cub ft, the one closed end being flat. Give the dimensions of the gasometer.

16. Find the volume of the greatest cylindrical parcel which may be sent by parcel post Given that the combined length and girth must not be greater than 6 feet.

17. Find the values of x which will make $\sin(x-a)\cos x$ a maximum or minimum.

18. Determine the maximum and minimum values of $f(x)$ when

$$f(x)=(x-2)^4(x-4)^2$$

19. Find the values of x which make $x(a-x)^2(2a-x)^3$ a maximum or minimum.

20 Find the least area of canvas that can be used to construct a conical tent whose cubical capacity is 800 cub. feet

21 Show that the maximum and minimum values of $y=\frac{x^2}{1+x^2}$ are $\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

22 The hypotenuse of a right-angled triangle is given; find the lengths of the other sides when the area is a maximum

23. Find the maximum and minimum values of

$$\text{(i) } x^3-6x^2+9x+10, \quad \text{(ii) } \sqrt{4a^2x^3-2ax^3}.$$

CHAPTER XVIII.

SUCCESSIVE DIFFERENTIATION. TAYLOR'S AND MACLAURIN'S THEOREMS.

Successive differentiation.—In the process of differentiation we have already found that when an expression contains x to any power, its differential contains x to a power lower by unity; we may consider such a differential of a function as a new function, and proceed to determine its differential

Ex. 1. Let $y=f(x)$, where $f(x)=3x^4$, $\frac{dy}{dx}=12x^3$

As the differential contains x^3 we may proceed to differentiate it as a new function The differential of $12x^3$ is $36x^2$, and is called the second differential of $f(x)$, and may be denoted by

$$\frac{d\left(\frac{dy}{dx}\right)}{dx}$$

This expression is more conveniently written in the usual form $\frac{d^2y}{dx^2}$

Repeating the process, the third differential $\frac{d^3y}{dx^3}=72x$ is obtained; and similarly, $\frac{d^4y}{dx^4}=72$ As this, the fourth, differential does not contain x, all succeeding differential coefficients will be zero

Care must be taken not to confuse $\frac{d^2y}{dx^2}$ with $\left(\frac{dy}{dx}\right)^2$. The former denotes the differential of the differential of y with respect to x, the latter is the square of the differential of y.

If u and v are functions of x, it can easily be shown that

$$\frac{d^n(uv)}{dx^n}=u\frac{d^nv}{dx^n}+n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}}+\frac{n(n-1)}{1\,.\,2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}}+\text{etc}+\frac{d^nu}{dx^n}v.$$

This is called the **Theorem of Leibnitz.**

If we differentiate $y=uv$, we find, as on p. 320,

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

Differentiating again, this becomes

$$\frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+\frac{du}{dx}\frac{dv}{dx}+\frac{dv}{dx}\frac{du}{dx}+v\frac{d^2u}{dx^2}$$

$$=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$$

In a similar manner, from the third differentiation,

$$\frac{d^3y}{dx^3}=u\frac{d^3v}{dx^3}+3\frac{du}{dx}\frac{d^2v}{dx^2}+3\frac{d^2u}{dx^2}\frac{dv}{dx}+v\frac{d^3u}{dx^3},$$

and, generally,

$$\frac{d^ny}{dx^n}=u\frac{d^nv}{dx^n}+n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}}+\frac{n(n-1)}{1\,.\,2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}}+\ldots+\frac{d^nu}{dx^n}v,$$

in which the coefficients follow the law of the Binomial Theorem. Hence the result follows.

The simplest case of successive differentiation occurs with the function $y=e^x$, in which all the differential coefficients are equal to the original functions.

Ex. 2. Let $y=ax^4$

Then $\quad \frac{dy}{dx}=4ax^3,\ \frac{d^2y}{dx^2}=12ax^2,\ \frac{d^3y}{dx^3}=24ax.$

A convenient notation is to write

$$y=f(x),$$

then $\quad \frac{dy}{dx}=f'(x),\ \frac{d^2y}{dx^2}=f''(x),$ etc.

Thus, if $\quad f(x)=ax^4,$ then $f'(x)=4ax^3,$

$$f''(x)=12ax^2,\ f'''(x)=24ax.$$

Ex. 3. Let $y = a \sin x$.

Then $$\frac{dy}{dx} = a \cos x = a \sin\left(x + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = \frac{d(a \cos x)}{dx} = -a \sin x = a \sin\left(x + 2 \cdot \frac{\pi}{2}\right)$$

Similarly, $$\frac{d^3y}{dx^3} = -a \cos x = a \sin\left(x + 3\frac{\pi}{2}\right).$$

Ex 4. Let $y = a \sin bx$.

$$\frac{dy}{dx} = ab \cos bx = ab \sin\left(bx + \frac{\pi}{2}\right),$$

$$\frac{d^2y}{dx^2} = -ab^2 \sin bx = ab^2 \sin\left(bx + 2\frac{\pi}{2}\right),$$

$$\frac{d^3y}{dx^3} = -ab^3 \cos bx = ab^3 \sin\left(bx + 3\frac{\pi}{2}\right),$$

and $$\frac{d^ny}{dx^n} = ab^n \sin\left(bx + n\frac{\pi}{2}\right).$$

Implicit functions.—So far we have confined our attention to functions in which y occurs alone on the left-hand side of the equation Such are called **explicit functions**; in contradistinction an **implicit function** is one in which the variable y is not expressed directly as a function of x. We proceed to show how to find the differential coefficient of such an expression. The method adopted may be seen from the following examples:

Ex. 5. $$2yx + ay^2 = bx^2. \qquad \text{(i)}$$

Differentiating according to x, we obtain

$$2y + 2x\frac{dy}{dx} + 2ay\frac{dy}{dx} = 2bx,$$

dividing by 2 and rearranging,

$$(ay + x)\frac{dy}{dx} = bx - y;$$

$$\therefore \frac{dy}{dx} = \frac{bx - y}{ay + x} \qquad \text{(ii)}$$

This equation admits of being reduced to a simpler form by using Eq. (i).

Thus, from (i), $$bx^2 - yx = ay^2 + yx,$$

or $$x(bx - y) = y(ay + x);$$

$$\frac{y}{x}=\frac{bx-y}{ay+x}.$$

Substitute this value in (ii), and we obtain

$$\frac{dy}{dx}=\frac{y}{x}$$

For verification (i) may be treated as a quadratic for y;

$$\therefore\ y=\frac{-x+\sqrt{x^2+abx^2}}{a}$$

$$=\frac{x}{a}(-1+\sqrt{1+ab});$$

$$\frac{dy}{dx}=\frac{1}{a}(-1+\sqrt{1+ab})=\frac{y}{x}.$$

Ex. 6 The equation

$$xy=c^2 \text{ or } y=\frac{c^2}{x} \quad \ldots \quad \text{(i)}$$

is known as the rectangular hyperbola;

$$\therefore\ \frac{dy}{dx}=-\frac{c^2}{x^2}.$$

Now, consider it as an implicit function, in which case we have, by differentiating both sides (xy being the product of two functions of x),

$$x\frac{dy}{dx}+y=0;$$

$$\frac{dy}{dx}=-\frac{y}{x}.$$

Substitute the value of y from (i), and we find as before that

$$\frac{dy}{dx}=-\frac{c^2}{x^2}.$$

Partial differentiation.—In the preceding example, in which the relation betweeen x and y may be denoted by $f(xy)=0$, the result obtained by differentiation is precisely the same as would be obtained by differentiating the given expression, firstly with regard to x assuming y to be constant, and secondly with regard to y assuming x to remain constant, and finally taking the quotient with the opposite sign.

The process of differentiating with respect to one only of two or more variables is known as **partial differentiation.** It is usually denoted by such symbols as $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, which read as the partial differential coefficient of $f(x, y)$ with respect to x, and a corresponding expression for y, or shortly, "the partial with respect to x," "the partial with respect to y."

Ex. 7 Let $f(x, y)=2yx+ay^2-bx^2=0$

Differentiating first with respect to x, keeping y constant, we find

$$\frac{\partial f}{\partial x}=2y-2bx$$

Next differentiating with regard to y, keeping x constant,

$$\frac{\partial f}{\partial y}=2x+2ay$$

In order to convert $\frac{\partial f}{\partial y}$, which is a differentiation with respect to y, into one with respect to x, we must multiply by $\frac{dy}{dx}$, or the differential coefficient of y with respect to x

Then, $$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=2y-2bx+2(x+ay)\frac{dy}{dx}=0,$$

or $$y-bx+(x+ay)\frac{dy}{dx}=0;$$

$$\therefore \frac{dy}{dx}=-\frac{y-bx}{x+ay}=\frac{bx-y}{ay+x}=\frac{y}{x},$$

or for all implicit relations between two variables such as x and y, we have

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0;$$

$$\therefore \frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

EXERCISES XL.

1. If $y=x^4+3x^3-x^2+5$, find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$

2. $y=\sin ax$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

3. $y = A \sin ax$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

4. $y = A \cos ax$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

5 $y = \sqrt{x^3}$, find $\frac{d^2y}{dx^2}$.

6 $s = v_0 t + \frac{1}{2}at^2$, find $\frac{d^2y}{dx^2}$.

7. $x = A \sin nt + B \cos nt$, prove that $\frac{d^2x}{dt^2} + n^2x = 0$.

8 $y = e^{-x} \cos x$, prove that $\frac{d^4y}{dx^4} + 4y = 0$.

9. $x = \theta^2 \log \theta$, prove that $\frac{d^3x}{d\theta^3} = \frac{2}{\theta}$.

10. $x = \tan \theta + \sec \theta$, prove that $\frac{d^2x}{d\theta^2} = \frac{\cos \theta}{(1 - \sin \theta)^2}$

11. Show by means of the following examples that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}:$$

(i) $u = \frac{x^2y}{a^2 - y^2}$; (ii) $u = x \sin y + y \sin x$

12 If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

13 If $y = \frac{\cos (2 \tan^{-1} x)}{1 + x^2}$, show that

$$(1 + x^2)\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$$

Maclaurin's Theorem.—It is frequently necessary to expand an algebraical or trigonometrical function into an infinite series. Examples are furnished in the expansion of a series by the Binomial Theorem, and various methods have already been given for the expansion of such expressions as $(a + x)^x$, e^x, a^x, $\log_e (1 + x)$, etc, in a series of ascending powers of x. We may now find a general theorem by means of which all the preceding, as well as others, may be expanded.

Let y denote some function of x, or $y = f(x)$. Assuming that this function, when expanded, can be represented by a series of ascending powers of x, whose coefficients A, B, C, ... do not contain x, we may write

$$y = f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Differentiating,

$$\frac{dy}{dx}=B+2Cx+3Dx^2+4Ex^3+\dots.$$

Differentiating again,

$$\frac{d^2y}{dx^2}=2C+3\cdot 2Dx+4\cdot 3Ex^2+\dots,$$

and $$\frac{d^3y}{dx^3}=2\cdot 3D+4\cdot 3\cdot 2Ex+\dots$$

Now, as the series must be true for all values of x, it must be true for the value $x=0$; and, therefore, if the expressions (y), $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, etc., denote the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc., for the particular case when $x=0$, we obtain

$$(y)=A, \quad \left(\frac{dy}{dx}\right)=B,$$

$$\left(\frac{d^2y}{dx^2}\right)=2C, \quad \left(\frac{d^3y}{dx^3}\right)=2\cdot 3D, \text{ etc.},$$

or $$A=(y), \quad B=\left(\frac{dy}{dx}\right), \quad C=\frac{1}{1\cdot 2}\left(\frac{d^2y}{dx^2}\right),$$

$$D=\frac{1}{1\cdot 2\cdot 3}\left(\frac{d^3y}{dx^3}\right), \quad E=\frac{1}{1\cdot 2\cdot 3\cdot 4}\left(\frac{d^4y}{dx^4}\right), \text{ etc.}$$

Substituting these values in Eq (i),

$$y=(y)+\left(\frac{dy}{dx}\right)x+\frac{1}{1\cdot 2}\left(\frac{d^2y}{dx^2}\right)x^2+\frac{1}{1\cdot 2\cdot 3}\left(\frac{d^3y}{dx^3}\right)x^3+,\dots \quad \text{(ii)}$$

or $$y=f(x)_0+xf'(x)_0+\frac{x^2}{1\cdot 2}f''(x)_0+\frac{x^3}{1\cdot 2\cdot 3}f'''(x)_0+,\quad \dots \text{(iii)}$$

in which the given function $y=f(x)$ is represented in a series of ascending powers of x with constant coefficients.

The result given by (ii) or (iii) is known as **Maclaurin's Theorem.**

If any function x be changed into $x+h$, then the differential coefficient will be the same whether we suppose x to vary uniformly and h to remain constant, or h to vary and x to remain constant.

It is an easy matter to see that this is so from a simple example as follows.

Let $$y=x^3.$$

Then, when x becomes $x+h$, we may write

$$y'=(x+h)^3.$$

On the supposition that x varies and h remains constant, we obtain

$$\frac{\partial y'}{\partial x}=3(x+h)^2.$$

Also if h varies and x is constant,

$$\frac{\partial y'}{\partial h}=3(x+h)^2,$$

$$\therefore \frac{\partial y'}{\partial x}=\frac{\partial y'}{\partial h}.$$

Taylor's Theorem.—A theorem of great importance, known as Taylor's Theorem, may now be stated.

Let $$y=f(x),$$

and let y' denote the new function when x becomes $x+h$,

$$y'=y+Ah+Bh^2+Ch^3+\ldots \quad \ldots \ldots \text{(i)}$$

whose coefficients A, B, C, etc., contain x but not h.

Differentiate on the supposition that x is constant and h varies;

$$\frac{\partial y'}{\partial h}=A+2Bh+3Ch^2+\ldots \quad \ldots \text{(ii)}$$

Next let x vary and h remain constant, then

$$\frac{\partial y'}{\partial x}=\frac{\partial y}{\partial x}+\frac{\partial A}{\partial x}h+\frac{\partial B}{\partial x}h^2+\text{etc.} \quad \ldots \ldots \text{(iii)}$$

As the left-hand sides of Equations (ii) and (iii) are equal, the two series are identical, and therefore the coefficients of the same powers of h are equal;

$$\therefore A=\frac{\partial y}{\partial x},\quad B=\frac{1}{2}\frac{\partial A}{\partial x},\quad C=\frac{1}{3}\frac{\partial B}{\partial x},\quad D=\frac{1}{4}\frac{\partial C}{\partial x},\ \text{etc.},\ \ldots$$

Substituting in B the value of A;

$$\therefore B=\frac{1}{2}\frac{\partial A}{\partial x}=\frac{1}{1.2}\frac{\partial^2 y}{\partial x^2}.$$

Similarly, $$C=\frac{1}{1.2.3}\frac{\partial^3 y}{\partial x^3}, \text{ etc.}$$

Now, substituting these values in (i),

$$y'=y+h\frac{\partial y}{\partial x}+\frac{h^2}{1.2}\frac{\partial^2 y}{\partial x^2}+\frac{h^3}{1.2.3}\frac{\partial^3 y}{\partial x^3}+\dots,$$

or, $$f(x+h)=f(x)+hf'(x)+\frac{h^2f''(x)}{1.2}+\frac{h^3f'''(x)}{1.2.3}+\dots, \qquad \text{(iv)}$$

where $f'(x)$, $f''(x)$, etc., refer to differentiation with respect to x only. This is **Taylor's Theorem.**

Ex. 1. Let $f(x)=x^n$, $f(x+h)=(x+h)^n$,

$$f'(x)=nx^{n-1},\ f''(x)=\frac{n(n-1)x^{n-2}}{1.2}, \text{ etc.};$$

$$(x+h)^n=x^n+nhx^{n-1}+\frac{n(n-1)h^2x^{n-2}}{1.2}+\dots,$$

the well-known binomial expansion.

Examples of the use of Taylor's Theorem.—A few examples of the use of Taylor's Theorem are given; others of a similar kind may easily be obtained if necessary.

Ex. 2. Given $\sin 30°=0.5$, find the value of $\sin 30° 30'$.

In this case h is the radian measure of $30'$;

$$30'=\frac{3.14159\times 30}{60\times 180}=0.0087$$

From Equation (iv), we find

$$f(x)=\sin 30°=0.5,\ f'(x)=\cos 30°=0.866,$$

$$f''(x)=-\sin 30°=-0.5,\ f'''(x)=-\cos 30°=-0.866$$

Substituting these values in Eq. (iv), we find

$$\sin 30° 30'=\sin 30°+0.0087\times\cos 30°+\frac{(0.0087)^2(-\sin 30°)}{1.2}+\text{etc.}$$

$$=0.5+0.0087\times 0.866-\frac{(0.0087)^2}{1.2}\times 0.5\ \dots+\text{etc}$$

$$=0.5+0.0075342-0.000018922=0.5075.$$

Development of $\log_e(1+x)$.—The development of this series has already been found (p. 293); it may also be obtained by Taylor's Theorem.

Ex. 3. Let $\qquad y=\log_e x,\ y'=\log_e(x+h),$

$$\frac{dy}{dx}=\frac{1}{x},\ \frac{d^2y}{dx^2}=-\frac{1}{x^2},\ \frac{d^3y}{dx^3}=\frac{2}{x^3},\ \frac{d^4y}{dx^4}=-\frac{2.3}{x^4}+\text{etc.}$$

Substituting these values in Taylor's Theorem, we obtain

$$\log_e(x+h)=\log_e x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\frac{h^4}{4x^4}+\ldots.$$

Substituting unity for x, and x for h, then, since

$$\log_e 1=0,$$

$$\log_e(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots,$$

the same result as that already found on p 293

Maclaurin's Theorem can easily be obtained from Taylor's Theorem, thus

$$f(x+h)=f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1\ 2}f''(x)+\ldots$$

Now put $x=0$, and for h write x, and we find

$$f(x)=f(x)_0+xf'(x)_0+\frac{x^2}{1\ 2}f''(x)_0+\ldots$$

The meaning attached to the symbols may be shown by $f''(x)_0$, which indicates that $f(x)$ is to be differentiated twice with respect to x, and finally put $x=0$ in the result.

Ex. 4. Expand the function $y=\sin x$ in a series of ascending powers of x

$y=\sin x$; $\quad\therefore$ when $x=0$, $(y)=0$, or $f(x)=0$.

$\dfrac{dy}{dx}=\cos x$; $\quad$ when $x=0$, $f'(x)=\cos 0=1$.

Also $\quad f''(x)=-\sin x$, $\quad$ when $x=0$; $\quad f''(x)=0$.

$\qquad f'''(x)=-\cos x$, $\quad$ when $x=0$; $\quad f'''(x)=-1$.

Substituting these values in (iii),

$$y=\sin x=x-\frac{x^3}{1\ 2.3}+\frac{x^5}{1.2\ 3.4\ 5}-\frac{x^7}{7!}+\ldots.$$

where $7!$ is read as *factorial seven*, and simply means $1\times2\times3\times4\times5\times6\times7$.

Similarly, if $y=\cos x$, we obtain

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots.$$

Infinitesimal quantities.—If we suppose some quantity denoted by Q to be divided into a large number of equal parts (say a million, 10^6), each of the equal quantities, Q_1, will be small in comparison with Q. Again, if we assume each of these equal parts to be again divided into one million, then each part would be $\frac{Q}{10^{12}}$ or $\frac{Q_1}{10^6}$.

Proceeding in this manner, any number of quantities may be obtained in which the second is extremely small in comparison with the first, the third in comparison with the second, and so on.

The quantities so obtained may be denoted by $Q_1, Q_2, Q_3, \ldots$, and are known as small quantities of the first, second, third, etc., orders. These small quantities are such that those of the second order may be neglected in comparison with those of the first order, etc. Examples of their use are furnished in calculating the numerical values of a given function from a series in which the numerical values of the several terms rapidly diminish.

Then, for example, since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \text{ and } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.$$

It follows that if x is of the first order, then $\sin x$ and $1 - \cos x$ are respectively of the first and second orders of small quantities.

Value of sin x.—The value of $\sin x$ can be obtained to any requisite degree of accuracy by using a few terms of the series (p. 381), viz.:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.$$

Thus, for an angle of 30°, as $\sin 30° = 0.5236$ radians, we obtain by substitution

$$\sin 30° = 0.5236 - \frac{(0.5236)^3}{6} + \frac{(0.5236)^5}{120}$$
$$= 0.5236 - 0.0239 + 0.0003 = 0.5000.$$

Thus, $\sin 30° = 0.5$.

Similarly, $$\sin 60° = 1.0472 - \frac{(1.0472)^3}{6} + \frac{(1.0472)^5}{120} - \frac{(1.0472)^7}{5040}$$
$$= 1.0472 - 0.1914 + 0.0105 - 0.0003 = 0.8660.$$

Calculation of the value of cos x.—The numerical value of the cosine of a given angle may also be determined by means of the appropriate series.

Ex. 5. Calculate the numerical value of cos 30°.

We have $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots,$$

and as $30° = 0{\cdot}5236$ radians, we obtain, by substitution,

$$\cos 30° = 1 - \frac{(0{\cdot}5236)^2}{2} + \frac{(0{\cdot}5236)^4}{24}$$
$$= 1 - 0{\cdot}1371 + 0{\cdot}0031 = 0{\cdot}8660.$$

In a similar manner other values may be calculated and the results compared with those in Table V.

Since $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{etc.},$$

we have, dividing by x,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \text{etc.};$$

$$\mathrm{Lt}_{x=0}\left[\frac{\sin x}{x}\right] = \mathrm{Lt}_{x=0}\left[1 - \frac{x^2}{3!} + \text{etc.}\right] = 1.$$

Small angles.—If the angle x be so small that x^2 and all succeeding terms can be omitted, then from the preceding series $\sin x = x$, or **when an angle is small, its sine is approximately equal to the circular measure of the angle and its cosine is approximately equal to unity**

Series for tan x.—The series for $\tan x$ may be obtained from Maclaurin's Theorem

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

Here $f(x) = \tan x$; $\therefore f(0) = 0.$

$f'(x) = 1 + \tan^2 x = 1 + [f(x)]^2$; $f'(0) = 1.$

$f''(x) = 2f(x)f'(x)$; $\therefore f''(0) = 0.$

$f'''(x) = 2f''(x)f(x) + 2[f'(x)]^2$; $\therefore f'''(0) = 2$, etc.

$f^{iv}(0) = 0,$ $f^{v}(0) = 16.$

Hence, by substitution,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

Maxima and minima.—We have already found (p. 356) that if, at a point denoting a maximum value on a curve, the abscissa x receives a small increment, the corresponding value of y is less than its preceding value. Or, the slope of the curve becomes negative. In other words, the tangent to a curve making a positive angle with the axis of x varies until at a point indicating a maximum value it becomes horizontal. When x is increased past this point, the inclination of the tangent is in a negative direction. For a minimum value the inclination of the tangent varies from negative through zero to positive

Thus, if $y=f(x)$, and $f(a)$ is a maximum or minimum value Then $f(a)$ will be a maximum value of $f(x)$ if $f'(x)$ changes from a positive to a negative value as x passes through a; and $f(a)$ will be a minimum value if $f'(x)$ changes from a negative to a positive value as x passes through a

Analytically, by **Taylor's Theorem**, let $y=f(x)$, and let x become $x+\delta x$, then, since $f'(x)=0$, we have

$$f(x+\delta x)=f(x)+\frac{f''(x)}{1 \cdot 2}(\delta x)^2+\ldots \qquad \text{(i)}$$

Also, if x becomes $x-\delta x$, we get

$$f(x-\delta x)=f(x)+\frac{f''(x)}{1 \cdot 2}(-\delta x)^2+\ldots$$

$$=f(x)+\frac{f''(x)}{1 \cdot 2}(\delta x)^2+\ldots \ldots \ldots \ldots \text{(ii)}$$

From (i) and (ii), we see that if the second term $f''(x)$ be positive, then in both expressions the values of the right-hand side of the expression is greater than $f(x)$ Therefore the value of the ordinate y diminishes in passing from the point $x-\delta x$ to the point x, and y is said to have a minimum value. Similarly, if $\frac{d^2y}{dx^2}$ or $f''(x)$ is negative, the value of y is a maximum. In this way obtain a rule which may be thus stated:

If $y=f(x)$, the value or values which denote a maximum or minimum are obtained by determining the value or values which make $f'(x)=0$ To ascertain whether the values obtained denote a maximum or a minimum, find the value of $\frac{d^2y}{dx^2}$ or

$f''(x)$ and substitute for x If the resulting value is negative, it corresponds to a maximum, and to a minimum if the value is positive.

Ex 1. Determine the values of x which make $x^3 - 6x^2 + 9x - 12$ a maximum or a minimum.

Let
$$y = x^3 - 6x^2 + 9x - 12,$$
$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

When y is a maximum or minimum,
$$\frac{dy}{dx} = 0\ ;$$
$$\therefore\ 3x^2 - 12x + 9 = 0, \qquad \text{(i)}$$
or
$$x^2 - 4x + 3 = 0 \text{ or } (x-1)(x-3) = 0\ ;$$
$$x = 1 \text{ or } 3$$

Differentiating (i), we obtain
$$\frac{d^2y}{dx^2} = 6x - 12\ ;$$

when $x = 1$, $\frac{d^2y}{dx^2} = -6$, a negative quantity, and therefore corresponds to a maximum

Similarly, when $x = 3$, $\frac{d^2y}{dx^2} = 6$, a positive quantity.

Hence, $x = 1$ corresponds to a maximum value.
and $x = 3$,, ,, minimum ,,

The rule may be stated thus —**An expression will be a maximum when the value of x, which makes $\frac{dy}{dx}$ zero, gives $\frac{d^2y}{dx^2}$ a negative sign, and a minimum when the value of x gives $\frac{d^2y}{dx^2}$ a positive sign.**

It will be noticed that this expresses only in a different form the rule already used in determining maxima and minima by plotting.

Points of inflexion.—It may happen that both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ vanish for the same value of x In that case, we do not necessarily have a maximum or a minimum Reference to Fig. 126 will show a point, H', where this is so.

If this happens $\frac{d^2y}{dx^2}$ should be differentiated again and the values that made $\frac{d^2y}{dx^2}$ vanish, as well as $\frac{dy}{dx}$ zero, should be substituted in $\frac{d^3y}{dx^3}$.

If after substitution the result is not zero, then the point considered is a point of inflexion; but if it is zero, then the differentiation must again be tried to find whether the new differential vanishes. The process must be repeated until we obtain a differential which does not vanish If the first non-vanishing coefficient is of **odd** order it is a **point of inflection**; if of **even** order it is a **turning point**, *i e* one that is either a maximum or a minimum

Ex. 2. Let

$$y=x^3-3x^2+3x-13,$$

$$\frac{dy}{dx}=3x^2-6x+3=3(x-1)^2.$$

which vanishes when $x=1$.

Differentiating again;

$$\frac{d^2y}{dx^2}=6(x-1),$$

and this is also zero when $x=1$;

$\therefore$ differentiate again, and obtain

$$\frac{d^3y}{dx^3}=6$$

But this result does not contain x, and is not zero when $x=1$, and therefore the point $x=1$ is neither a maximum nor a minimum, but a point of inflexion; that is to say, the tangents to the curve on either side of the point are inclined in the same (positive or negative) direction to the tangent at the point itself.

Ex. 3. Find the maximum and minimum values (if any) of y when

$$y=x^4-8x^3+24x^2-32x,$$

$$\frac{dy}{dx}=4(x^3-6x^2+12x-8)=4(x-2)^3,$$

and this is zero when $x=2$.

Hence, differentiating again,

$$\frac{d^2y}{dx^2}=12(x-2)^2, \text{ and this is zero when } x=2.$$

Again, $\frac{d^3y}{dx^3}=24(x-2)$, and this is zero when $x=2$.

$$\frac{d^4y}{dx^4}=24.$$

In this case the first differential which does not vanish is of even order.

Thus $x=2$ gives a minimum value of y.

Each of these cases, Exs. 2 and 3, can be simplified; the first, by the substitution of z for $x-1$, becomes $y=z^3-12$.

The second, by putting $x-2=v$, becomes $y=v^4+16$.

In each case the resulting expression may be treated in the usual manner.

EXERCISES. XLI.

1 Expand $\log_e(1+x)$.

2. Expand as far as x^4.

(i) $\log(x+\sqrt{x^2+a^2})$; (ii) $(e^x+e^{-x})^n$

3 Expand, by Maclaurin's Theorem, $\tan^4 x$ in terms of x to three terms

Find the first and second differential coefficients of the following:

4 x^x. 5. $xe^{\tan x}$. 6. $\tan^{-1}x$.

Find the n^{th} differential coefficients of:

7. $x^2\log x$. 8. x^3e^x.

9 Expand $\tan^{-1}x$ in a series of ascending powers of x by Maclaurin's Theorem.

10 Expand $\sin^{-1}(x+h)$ to three terms by Taylor's Theorem.

11. Expand $e^x\log_e(1+x)$ by Maclaurin's Theorem

12. Expand $\sin x$ in terms of x by Maclaurin's Theorem to three terms.

CHAPTER XIX.

INTEGRATION.

Integration.—We may consider integration as the inverse process of differentiation. Thus, for example, from a relation connecting x and y, the process by which $\frac{dy}{dx}$ is obtained is called **differentiation.** Conversely, given a differential expression, the previous process may sometimes be reversed and the **integral** obtained, the object being to determine the expression, or function, from which the given differential expression has been obtained. We are able in this way, to write down, in many cases, the original expression by mere inspection Or, we may make use of a rule which is readily seen from the corresponding rule in differentiation.

Ex. 1. Thus, if $y=x^3$,

$$\frac{dy}{dx}=3x^2$$

This may be written in the form

$$dy=3x^2dx.$$

Integrating, $$\int dy=\int 3x^2dx,$$

or $$y=x^3.$$

These, and similar expressions, may be obtained by using the following rule

To find the integral of a power of x, add unity to the index and divide by the index thus increased.

[As any constant quantity connected with a function by a positive, or negative, sign (indicating addition or subtraction) disappears during differentiation; therefore, a constant, which

may conveniently be denoted by C, must be added after integration; its value is determined from the conditions of the given problem.]

An important exception to this rule is furnished when $n=1$, or the quantity to be integrated is $\frac{1}{x}$ This will, however, be recognized to be the inverse of the differentiation of $\log x$.

Thus, if $\quad y=\log x, \quad \frac{dy}{dx}=\frac{1}{x};$

and if $\quad y_1=\frac{1}{x}, \quad \int y_1 dx=\log x$

The symbol $\int y_1 dx$ is read as "**the integral of y_1 with respect to x.**"

As illustrations of the meaning of integration consider the two progressions, arithmetical and geometrical.

The integral as the sum of a series in arithmetical progression.—In the series

$$a^2+2a^2+3a^2+\quad+na^2, \quad \ldots\ldots \quad \ldots\ldots \text{(i)}$$

the sum of n terms is $\frac{a^2\times n(n+1)}{2}$ (p 269).

Now, as the number of terms may be of any magnitude, it is possible, if a is altered inversely as n, to make na always the same, say equal to x

Thus, the sum becomes $\frac{x(x+a)}{2}$. . . . .(ii)

Now, as na is to remain constant, it follows that as n becomes greater and greater, a becomes less and less, and eventually, when a becomes zero, the sum of the series from (ii) is $\frac{x^2}{2}$.

We may with advantage rewrite the original series and put δx instead of a.

$$(\delta x)^2+2(\delta x)^2+\ldots+n(\delta x)^2.$$

But, as before, $n\delta x=x$ Thus, we find

$$(\delta x)^2+\ldots+x\delta x.$$

This is obviously an ordinary arithmetical progression, and the sum of the series, which may be denoted by Σ, gives

$$\Sigma\{n(\delta x)^2\}=\Sigma\{x\,\delta x\}=\frac{(\delta x)^2 n(n+1)}{2}=\frac{x(x+\delta x)}{2}$$

by the usual formula

So long as δx is assumed to be of any magnitude, the sum may be found by the preceding expression. If, now, we assume δx to be zero, we may write the integration sign $\int$ instead of the summation sign Σ, and also dx instead of δx, and we obtain

$$\int x dx=\frac{x^2}{2}.$$

It may perhaps be easier to follow this proof if the various steps are interpreted graphically For this purpose, take in the usual manner, two perpendicular axes, mark off horizontally distances equal to a, and vertically distances b, $2b$, $3b$, ... nb, as in Fig. 135.

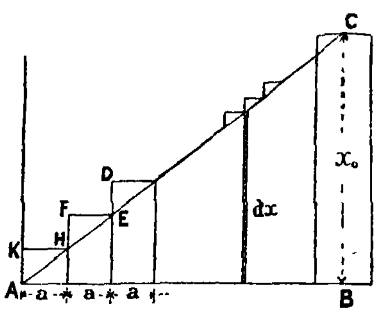

FIG 135.

The area of the first rectangle, its two sides AK and KH, is ab, or the first term of the given series. Similarly, the area of the second rectangle represents the second term, and so on, the last term being nab. AB is equal to na, and the assumption that na is constant implies that AB and BC are to be constant lengths; the sum of the series is the sum of all the rectangles into which the figure may be assumed to be divided, and is equal to the area of the triangle ABC together with the area of n half squares.

$$\therefore \text{ sum of series}=\frac{AB\times BC}{2}+\frac{nab}{2}$$

$$=\frac{n^2ab+nab}{2}=\frac{n(n+1)ab}{2},$$

i.e. the ordinary summation formula.

Now assume n to become very great. Then, since AB is to remain the same, the length a must be very small, and the size of the half squares will become very small.

Finally, when a is made indefinitely small, the corners of the squares will all lie on the line AC, and the sum of the series will be the area of $ABC=\frac{n^2ab}{2}$.

But, in the preceding case, na was denoted by x_0 and also $a=b$,

$$\frac{n^2a^2}{2}=\frac{x_0^2}{2}=\int x\,dx,$$

where x is 0 for the first term and x_0 for the last

Geometrical progression.—Consider the geometrical series

$$a+ar^a+ar^{2a}+ar^{3a}+\ldots+ar^{ma-1}.$$

The sum $=\frac{a(r^{ma}-1)}{r^a-1}$.

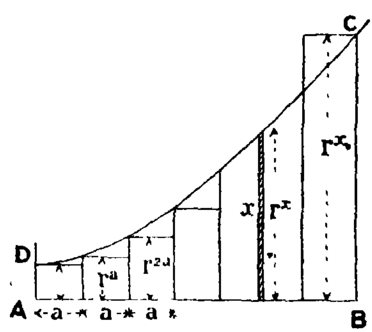

FIG 136.

As in the previous solution, we may represent the sum graphically

The area of the stepped figure (Fig 136) gives the sum of the series. Now, make $r^{ma}=\text{const}=r^{x_0}$,

$$ma=\text{const}=x_0.$$

Next assume m to become indefinitely large, and a very small. The steps in the curve (Fig 136) will disappear and a continuous curve will be obtained. Also r^{ma-1} will be very nearly equal to the next ordinate to it, or, in other words,

$$r^{ma-1}=r^{x_0};$$

$$\text{sum of series}=\frac{a(r^{x_0}-1)}{r^a-1}.$$

Expand r^a by the exponential theorem (p 292)

$$r^a=1+a\log_e r+\frac{a^2\log_e{}^2 r}{2\,!}+\text{etc.};$$

$$\therefore\ \text{sum of series}=\frac{r^{x_0}-1}{\log_e r+\frac{a\log_e{}^2 r}{2\,!}+\text{etc.}}$$

Now make a zero; that is, make the curve continuous, and the sum of the series becomes

$$\frac{r^{x_0}-1}{\log_e r}$$

If for a we substitute dx, the series will be written

$$dx+r^{dx}dx+r^{2dx}dx+\ldots+r^{ndx}dx,$$

or $\int r^x dx$, where the value of x is 0 at the beginning, and x_0 at the end of the series

The differential coefficient of $\frac{r^x}{\log_e r}$ is equal to $r^x dx$;

$$\int r^x dx=\frac{r^x}{\log_e r}+C.$$

As already indicated, added constants disappear during the process of differentiation, and therefore it is necessary in all cases to add a constant after integration.

At the beginning $x=0$, and, since no terms are included, the area is zero;

$$\therefore \text{ the value of the integral is } \frac{1}{\log_e r}+\text{const.}=0,$$

$$\text{const}=-\frac{1}{\log_e r}.$$

At the end of the series $x=x_0$ and $r^x=r^{x_0}$, and the value of the integral is $\frac{r^{x_0}}{\log_e r}+\text{const}=\frac{r^{x_0}}{\log_e r}-\frac{1}{\log_e r}=\frac{r^{x_0}-1}{\log_e r}$

This is the result obtained by the preceding method

The reason for the subtraction appears from the fact that only a small portion of the curve is used in order to obtain the resultant area, but by continuing the geometrical series backwards, as $\ldots+ar^{-2a}+ar^{-a}+a+ar^{a}+ar^{2a}+\ldots$, the curve would gradually reach the axis of x The unknown constant is the area between this produced part of the curve and the axes of x and y.

The operation just performed is called **integration between limits**, and when the area $ABCD$ is required, it is necessary to obtain **the integral of $r^x dx$ between the limits $x=0$ and $x=x_0$**

This is written as $\int_0^{x_0} r^x dx.$

The rule for such an integration is **first find the general integral**, *i.e* in this case $\frac{r^x}{\log_e r}$, **then subtract the result of substituting the lower limit in the integral from the result of substituting the upper limit.** In this case 0 and x_0 respectively;

$$\therefore \int_0^{x_0} r^x dx = \left[\frac{r^x}{\log_e r}\right]_0^{x_0} = \frac{r^{x_0}}{\log_e r} - \frac{1}{\log_e r}$$

$$= \frac{r^{x_0} - 1}{\log_e r}.$$

These examples show three distinct methods which may be used to find the integral of a given function.

(*a*) **By the summation of a series in which the terms alter gradually.**

(*b*) **By the process of finding an area.**

(*c*) **By inverting the process of differentiation.**

Obviously the result of integrating a given function by each of the methods should be identical, but it should be noticed that the first method is frequently impossible, the last two (*b*) and (*c*) are those in general use

It will be noticed that the first two methods are identical, the character of (*a*) is algebraical, that of (*b*) is graphical. But as there are many series the algebraical terms of whose sum is unknown, or, useless from this point of view, we are practically restricted to (*b*) and (*c*)

To obtain the general connection between integration, regarded as the inverse process of differentiation and obtaining an area, we may proceed as follows.

If $y=f(x)$, then when the form of the function is known, the process of differentiation can be carried out by the methods already described, and we obtain $\frac{dy}{dx}=f'(x)$, which is, in general, some other function of x.

Now, plot $y=f(x)$ and x, and make x and y zero together; also plot $\frac{dy}{dx}=f'(x)$ and x.

If $\frac{\delta y}{\delta x}$ is plotted instead of $\frac{dy}{dx}$, we should obtain a stepped curve, as indicated by the dotted lines (Fig. 137). The area of the rectangle M_1N_1RQ is $\frac{\delta y}{\delta x} \times \delta x$.

Now, y is the sum of all the small increments, from the place where $x=0$ to the place where $x=ON_1$,

i.e.
$$\sum_{x=0}^{x=ON_1} \delta y = \sum_{x=0}^{x=ON_1} \frac{\delta y}{\delta x}\delta x, \quad \ldots (\text{i})$$

and this is obviously equal to the

area of the stepped figure OCP_1N_1 . . (ii)

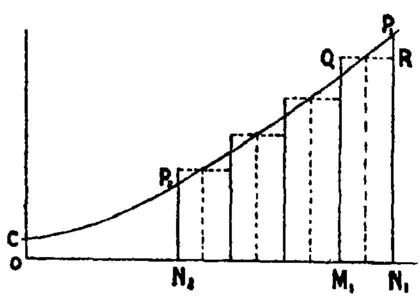

FIG 137

Hence, make δx indefinitely small, and $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$, and we write dx instead of δx, and $\int_0^{ON_1} \frac{dy}{dx}dx$ instead of (i)

Or, write $f'(x)$ for $\frac{dy}{dx}$, and the preceding becomes

$$y = \int_0^{ON_1} f'(x)dx = \text{area of figure } OCP_1N_1,$$

the steps having disappeared

Similarly, the area OCP_2N_2 is $\int_0^{ON_2} f'(x)\,dx = y_1$.

Hence, $y - y_1 =$ area OCP_1N_1 minus area OCP_2N_2, or the area $P_1N_1N_2P_2 = y - y_1 = \int_0^{ON_1} f'(x)dx - \int_0^{ON_2} f'(x)dx$.

This may be written.—**The increase in the value of the integral, as x increases from the value ON_2 to the value ON_1, is equal to the area between the curve $f'(x)$, the ordinates at N_2 and N_1, and the axis of x, and is equal to the value obtained by the inverse process of differentiation, finally substituting in the result the extreme values of x, and subtracting.**

For convenience, $\int^{ON_1} f'(x)dx - \int^{ON_2} f'(x)dx$ is written in the form $\int_{ON_2}^{ON_1} f'(x)dx$.

Ex. 2. Let $$f'(x) = \frac{x}{2}.$$

As on p. 390, $f(x) = \int f'(x)dx = \frac{x^2}{4} + \text{const.}$

Plot $f(x)$. This is a straight line passing through the origin (Fig 138). The area enclosed by the line, an ordinate at any point, and the axis of x can be obtained. Thus the area enclosed up to point P is $50 \times 25 \div 625$. Draw a straight line $A_1B_1C_1$ parallel to the axis of x and at any convenient distance from it. Make $A_1P_1 = 625$ (altering the vertical scale for the purpose). Again at a point on the curve where $x = 150$, $y = 75$; area enclosed $= 150 \times 75 \div 2 = 5625$. Proceeding in this manner, any number of points on a curve may be obtained, the ordinates of the curve denoting the area enclosed by the line up to the ordinate passing through the point.

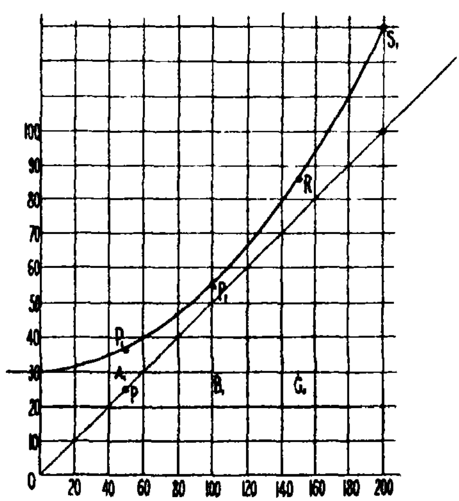

FIG 138.

Drawing the curve through the points P_1, R_1, S_1, we obtain the parabola $\frac{x^2}{4} + \text{const.}$

It will be seen that the value of the constant is unknown, because by moving the curve $O_1P_1Q_1R_1S_1$ parallel to the axis of y we do not alter the slope at the points, and therefore the shape of the curve remains the same. The constant must therefore be determined otherwise.

If, however, the difference in height between P_1 and R_1 is required, then $R_1C_1 - P_1A_1$ is the value required; and the result is obviously independent of the constant.

As in preceding case the value is

$$\int_{50}^{150} \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_{50}^{150} = \frac{150^2}{4} - \frac{50^2}{4} = 5000$$

From the diagram this is seen to be the area of $APRC$

Ex. 3. Let $f'(x) = \frac{1}{x} \times 0\ 434$

Then, as on p. 389, $\quad f(x) = 0\ 434 \int \frac{dx}{x}$

$$= 0{\cdot}434 \log_e x + \text{const.}$$

But $\quad 0\ 434 \log_e x = \log_{10} x$;

$$f(x) = \log_{10} x + \text{const}$$

Substituting values 0 5, 1, 1 2, etc., corresponding values of $f'(x)$ can be obtained. A few values are given in the following table

x	0 5	1·0	1·2	1·4	1 6	1 8	2 0
$f'(x) = 0{\cdot}434 \frac{1}{x}$	0 868	0 434	0·362	0 310	0·271	0·241	0 217

Now $\quad 0{\cdot}434 \int_1^2 \frac{1}{x}\,dx = 0{\cdot}434 \log_e 2 - 0\ 434 \log_e 1.$

But the logarithm of 1 to any base is zero, and

$$0{\cdot}434 \log_e 2 = \log_{10} 2 = 0\ 301\ ;$$

which is the area enclosed between the curve, the axis of x, and the ordinates $x=1$ and $x=2$

This result may be readily verified by drawing the curve on squared paper to a fairly large scale, and adding up the whole squares and partial squares enclosed by the curve.

Similarly, the logarithm of 2·5 is the area enclosed by the curve from $x=1$ to $x=2{\cdot}5$, and so on for the logarithm of any number.

Ex. 4. Integrate $\int \frac{dx}{x}$.

The indefinite integral is $\log_e x + C$.
Hence, if the limits are a and b,

$$\int_a^b \frac{dx}{x} = \log_e b - \log_e a.$$

Ex 5 Find the value of $\int_{\frac{1}{3}}^{1} \frac{dx}{x}$

$$\int_{\frac{1}{3}}^{1} \frac{dx}{x} = [\log x]_{\frac{1}{3}}^{1} = \log_e 3 = 0{\cdot}4771 \times 2{\cdot}3026 = 1{\cdot}0987.$$

Ex. 6 Find the value of $\int_1^4 x^2 dx$.

$$\int_1^4 x^2 dx = \tfrac{1}{3}[x^3]_1^4 = \tfrac{1}{3}(4^3 - 1^3) = 21$$

Ex 7 Show that

$$c\int_a^b x^n dx = \frac{c}{n+1}(b^{n+1} - a^{n+1})$$

Area of segment of a parabola.—In the curve

$$y = ax^2, \quad \ldots \quad \text{(iii)}$$

$$A = \int ax^2 dx = \frac{a}{3}x^3 + C \quad \ldots \quad \text{(iv)}$$

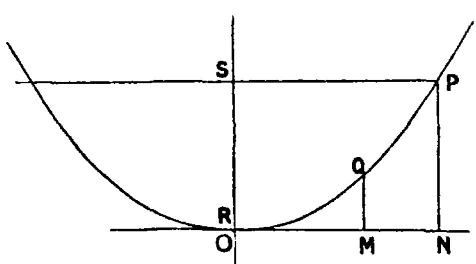

FIG 139 —Area of a parabola

The area $ONPR$ (Fig. 139) from $x=O$ to $x=ON$ (Fig. 139) is given by

$$A = \frac{a}{3} \times ON^3 \quad \ldots \quad \text{(v)}$$

We may eliminate the constant a by substituting in (iv) the values of x and y for any point, such as P.

Thus, $$PN = a \times ON^2;$$

$$\therefore a = \frac{PN}{ON^2}.$$

Substituting in (v), we obtain

$$A = \frac{PN}{3 \times ON^2} \times ON^3 = \tfrac{1}{3} \,.\, PN \times ON.$$

Hence, **the area of ONPR is one-third the area of the rectangle ONPS.** As there are, for each value of y, two values of x, it follows that the area of the segment of the parabola PSO is $\frac{2}{3}$ that of the rectangle $SONP$, an important result

Denoting OM by x_2, and ON by x_1, then the area of $MNPQ$ is given by

$$A = \int_{x_2}^{x_1} ax^2 dx = \frac{a}{3}\left(x_1^3 - x_2^3\right) \quad . \quad \ldots\ldots\ldots \text{(vi)}$$

A result from which, when the numerical values of a, x_1, and x_2 are known, the value of A can be obtained

Integration of sum of functions.—When differentiating an expression containing a number of distinct functions connected by the signs plus or minus, it was only necessary to differentiate each singly and obtain the algebraical sum of the differential coefficients.

In a similar manner when it is required to integrate an expression consisting of the sum of any number of functions, the integral of each separate term may be found, the sum of these separate integrals will be the integral required.

Ex. 8. Show that $\int (2x + x^2 - 1)dx = x^2 + \frac{1}{3}x^3 - x + C.$

Ex. 9 $$\int \left(ax^2 + \frac{1}{2\sqrt{x}}\right) dx = \frac{ax^3}{3} + \tfrac{1}{2}x^{\frac{1}{2}} \times 2 + C$$
$$= \frac{ax^3}{3} + \sqrt{x} + C.$$

We have already found that any constant which is a multiplier or divisor of a given function is a multiplier or divisor of the differential, hence a constant multiplier or divisor

following the integration sign may be removed and placed in front of the sign of integration.

The following list of some of the simpler functions and their differential coefficients will be found very useful, the list may easily be extended if necessary; it will be obvious that from such a list the integral of any function agreeing with any of the tabulated or known differential coefficients can be at once written down.

In all the following cases the constant of integration should be added.

If $y=x^n$, $\quad \dfrac{dy}{dx}=nx^{n-1}$; $\quad \therefore \int nx^{n-1}$ is x^n.

Hence, from the preceding,

$$\int x^n dx=\frac{x^{n+1}}{n+1},$$

$$y=\log x, \quad \frac{dy}{dx}=\frac{1}{x}; \quad \therefore \int \frac{1}{x}dx=\log x,$$

$$y=e^x, \quad \frac{dy}{dx}=e^x, \quad \int e^x dx=e^x,$$

$$y=\sin x, \quad \frac{dy}{dx}=\cos x, \quad \int \cos x dx=\sin x,$$

$$y=-\cos x, \quad \frac{dy}{dx}=\sin x, \quad \int \sin x dx=-\cos x,$$

$$y=\tan x, \quad \frac{dy}{dx}=\sec^2 x, \quad \int \sec^2 x dx=\tan x,$$

$$y=\sin^{-1}x, \quad \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}, \quad \int \frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x,$$

when $x<1$;

$$y=\tan^{-1}x, \quad \frac{dy}{dx}=\frac{1}{1+x^2}, \quad \int \frac{1}{1+x^2}dx=\tan^{-1}x,$$

$$y=\sec^{-1}x, \quad \frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}, \quad \int \frac{1}{x\sqrt{x^2-1}}dx=\sec^{-1}x$$

Thus, if $\qquad y=bx^n+C,$

the differential coefficient is

$$\frac{dy}{dx}=nbx^{n-1}, \text{ or } dy=nbx^{n-1}dx.$$

Hence, reversing the process, we see that the integral of $nbx^{n-1}dx$, written $nb\int x^{n-1}dx$,

$$=bx^n+C. \tag{1}$$

Ex. 10
$$\tfrac{1}{2}\int x^{-\frac{1}{2}}dx=\frac{\frac{1}{2}x^{-\frac{1}{2}+1}}{\frac{1}{2}}=x^{\frac{1}{2}}+C$$

Indefinite integral.—The expression (1) is called an **indefinite integral** of the function nbx^{n+1}, the value of the constant C being unknown. In practical applications the value of the constant and the integral can usually be determined from the conditions of the problem.

The preceding integrals are important and should be committed to memory. The following may be reduced to the preceding forms by one or more simple substitutions and rearrangements.

If
$$y=-\frac{1}{a}\cos ax,\quad \frac{dy}{dx}=\sin ax\ ;$$

$$\int\sin ax\,dx=-\frac{1}{a}\cos ax$$

The result may also be obtained as follows.

Ex. 11. Integrate $\sin ax\,dx$

Let $ax=z$, then $\dfrac{dz}{dx}=a$, or $dx=\dfrac{dz}{a}$;

$$\int\sin ax\,dx=\frac{1}{a}\int\sin z\,dz$$

But, from the preceding table,

$$\int\sin z\,dz=-\cos z\ ,$$

$$\frac{1}{a}\int\sin z\,dz=-\frac{1}{a}\cos z=-\frac{1}{a}\cos ax$$

In many cases an integration may be readily effected by means of a suitable simple substitution.

Ex. 12. Find the value of

$$\int \frac{1}{(a+bx)^n}\,dx$$

Put $a+bx=z$, then $dx=\frac{1}{b}dz$;

$$\int \frac{1}{(a+bx)^n}\,dx=\frac{1}{b}\int \frac{dz}{z^n}$$
$$=\frac{1}{b}\,\frac{z^{1-n}}{1-n};$$

replace z by $a+bx$;

$$\int \frac{1}{(a+bx)^n}\,dx=\frac{1}{b(1-n)}\,\frac{1}{(a+bx)^{n-1}}.$$

Ex 13 Integrate $\int e^{ax}dx$

Let $ax=z$, then $dx=\frac{dz}{a}$,

and $\int e^{ax}dx=\frac{1}{a}\int e^{z}dz$

$$=\frac{1}{a}e^{z}=\frac{1}{a}e^{ax}.$$

In some cases an integration may be effected by more than one method, the results obtained, although perhaps differing in appearance, may by suitable simplification be reduced to the same form Two such methods are used in the following examples

Ex 14. Find the integral of

$$\frac{x^3dx}{x^2-3x+2}.$$

It will be noticed that the numerator contains x to a higher power than the denominator In such a case it is necessary to divide the numerator by the denominator until the numerator contains x to a lower power than the denominator.

Thus $$\frac{x^3}{x^2-3x+2}=x+3+\frac{7x-6}{x^2-3x+2};$$

$$\int \frac{x^3}{x^2-3x+2}dx=\int\left\{x+3+\frac{7x-6}{x^2-3x+2}\right\}dx$$
$$=\frac{x^2}{2}+3x+\int \frac{7x-6}{x^2-3x+2}dx.$$

Resolving $\frac{7x-6}{x^2-3x+2}$ into its partial fractions, p 6, we obtain

$$\frac{7x-6}{x^2-3x+2}=\frac{8}{x-2}-\frac{1}{x-1},$$

$$\int\frac{7x-6}{x^2-3x+2}dx=8\log(x-2)-\log(x-1).$$

Hence $$\int\frac{x^3}{x^2-3x+2}dx=\frac{x^2}{2}+3x+8\log(x-2)-\log(x-1)$$

Instead of the preceding solution we could write the given integral as follows

$$\int\frac{x^3}{x^2-3x+2}dx=\frac{x^2}{2}+3x+\int\left\{\frac{7}{2}\left(\frac{2x-3}{x^2-3x+2}\right)+\frac{9}{2}\left(\frac{1}{x^2-3x+2}\right)\right\}dx$$
$$=\frac{x^2}{2}+3x+\frac{7}{2}\int\frac{2x-3}{x^2-3x+2}dx+\frac{9}{2}\int\frac{1}{x^2-3x+2}dx,$$

put $x^2-3x+2=z$,

$$\frac{dx}{dz}=2x-3,$$

also $\frac{7}{2}\int\frac{2x-3}{x^2-3x+2}dx$ becomes $\frac{7}{2}\int\frac{1}{z}dz=\frac{7}{2}\log z.$

Similarly $$\frac{9}{2}\int\frac{1}{(x-2)(x-1)}dx=\frac{9}{2}\int\left\{\frac{1}{x-2}-\frac{1}{x-1}\right\}dx$$
$$=\frac{9}{2}\log(x-2)\quad\frac{9}{2}\log(x-1)$$

Collecting the terms we find

$$\int\frac{x^3}{x^2-3x+2}dx=\frac{x^2}{2}+3x+\frac{7}{2}\log(x^2-3x+2)+\frac{9}{2}\log(x-2)-\frac{9}{2}\log(x-1).$$

This result appears to differ from the previous one, but

$\frac{7}{2}\log(x^2-3x+2)$ may be written $\frac{7}{2}\log(x-2)+\frac{7}{2}\log(x-1)$;

$$\therefore\int\frac{x^3}{x^2-3x+2}dx=\frac{x^2}{2}+3x+\frac{7}{2}\log(x-2)+\frac{7}{2}\log(x-1)+\frac{9}{2}\log(x-2)$$
$$-\frac{9}{2}\log(x-1)$$
$$=\frac{x^2}{2}+3x+8\log(x-2)\quad\log(x\quad 1)$$

Ex. 15. If $pv^s = c$ where c is a constant, find

$$\int p\,dv.$$

Here it is necessary to express p in terms of v

Thus, since $$pv^s = c,$$
then $$p = cv^{-s};$$

substitute for p; $$\int p\,dv = c\int v^{-s}dv$$
$$= c\frac{v^{-s+1}}{-s+1}$$

Thus, let $s = 0{\cdot}8$; then, $p = cv^{-0\cdot8}$;

$$c\int v^{-0\cdot8}dv = \frac{cv^{0\cdot2}}{0\cdot2} = 5cv^{0\cdot2}.$$

Let $s = 1$, then $$c\int v^{-1}dv = c\int\frac{1}{v}dv$$
$$= c\log_e v$$

Ex. 16 The rate (per unit increase of volume) of the reception of heat by a gas is h, p is its pressure, and v its volume, and c is a known constant

If $$pv^s = c, \tag{i}$$
s and c being constant, find h where

$$h = \frac{1}{\gamma - 1}\left\{v\frac{dp}{dv} + \gamma p\right\} \tag{ii}$$

If h is always 0, find what s must be.

From (i) $$p = cv^{-s}, \tag{iii}$$

$$\frac{dp}{dv} = -scv^{-s-1}$$

Substituting this value in (ii),

$$h = \frac{1}{\gamma - 1}\{v(-scv^{-s-1}) + \gamma p\};$$

$$\therefore\ h = \frac{1}{\gamma - 1}\{-scv^{-s} + \gamma p\}. \tag{iv}$$

Substituting from (iii) in (iv), we obtain

$$h = \frac{1}{\gamma - 1}(-sp + \gamma p),$$

when $h = 0$ we have $$\frac{p(\gamma - s)}{\gamma - 1} = 0;$$

giving $$s = \gamma.$$

Automatic integration.—Many instruments are in use by which integration is performed automatically. Familiar examples are furnished by meters of various kinds, such as gas and water meters Thus, assume an orifice, or tap, in connection with a water meter, then, if v denotes the velocity of the issuing water, the quantity which flows in a time t may be denoted by Q, where $Q=\int_0^t vdt$. This quantity is duly registered on the dial in front of the meter In a similar manner, the dials of a gas meter record the number of cubic feet of gas which passes through the meter

It must not be inferred that it is possible to integrate any given algebraic expression for some, as $\int\frac{dx}{\sqrt{(1+x^5)}}$, $\int\frac{dx}{\sqrt{(x^2+3)}}$ and others, have only been obtained by approximate methods. Thus, the form of the differential cannot always be derived from an algebraical expression In such a case the method adopted is to obtain an approximate value by the aid of series, etc.

Approximate methods.—In practical cases, such as finding the area or volume of an irregular figure, it frequently happens that the value of a definite integral cannot be obtained, and some approximate method must replace a more accurate integration Hence, it becomes necessary to ascertain what formulae may be used for the purpose

There are several methods by which, when numerical values of x and y are known, an approximate value of $\int_a^b ydx$ can be found. Of these the following are important

Simpson's Parabolic Rules, viz, the *one-third* and the *three-eighths* rules.

Weddle's rule, the **trapezoidal** and the **mid-ordinate** rules

Simpson's First Rule.—This important rule, also called Simpson's *one-third* rule, may be used when values of the ordinates of a given area, or volume, at equal distances are known, and when there is an *odd* number of such ordinates. It may be written in the form

$$\Sigma=\frac{s}{3}(A+4B+2C);$$

where Σ denotes the area when the ordinates are linear, and volume when the ordinates denote areas; s is the common distance between the ordinates, A the sum of the end ordinates, B the sum of the even ordinates, and C the sum of the odd

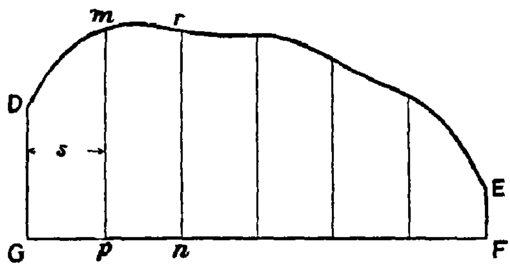

FIG 140

ordinates. If, as in Fig 140, there are seven ordinates, then the rule may be written

$$\text{Area } DGFE=\int_0^6 f(x)dx=\frac{s}{3}\{y_0+y_6+4(y_2+y_4+y_6)+2(y_3+y_5)\}.$$

Simpson's Second Rule.—Simpson's second or *three-eighths rule*, may be used when there are an *even* number of ordinates

$$A=\int_0^6 f(x)dx=\frac{3s}{8}\{y_0+y_5+2y_3+3(y_1+y_2+y_4+y_5)\}$$

Weddle's Rule.—This rule is applicable when there are 7 equidistant ordinates, and on the assumption that the boundary is a continuous curve the results are probably more accurate than those obtained by Simpson's Rules. The rule may be stated as follows

$$A=\frac{3s}{10}\{y_0+y_2+y_3+y_4+y_6+5(y_1+y_3+y_5)\}$$

Trapezoidal Rule.—The so-called trapezoidal rule is usually more easily manipulated than the preceding formulae, but the results are not so accurate as those obtained by Simpson's and Weddle's rules. The rule for 7 ordinates may be stated as follows:

$$A=s\{\tfrac{1}{2}(y_0+y_6)+y_1+y_2+y_3+y_4+y_5\}.$$

Mid-ordinate Rule.—If h is the mean ordinate of an irregular figure $DEFG$ (Fig. 141), then the product of h and the length DE is the area of the figure

The base DE is divided into a number of equal parts, and at the mid-point of each, as indicated by the dotted lines, perpendiculars are drawn; the sum of all such ordinates divided by the number of ordinates is the mean ordinate required. The approximation approaches nearer and nearer to the actual value as the number of ordinates is increased

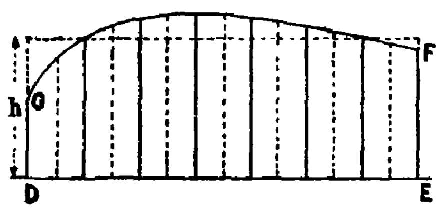

Fig 141

The sum of the ordinates is readily obtained by using a strip of paper and marking off a length equal to the first, and at the end of the first a length equal to the second, etc

Ex. 17 Find the area of the curve $y=x^2$, between the values $x=1$ and $x=7$.

Let A denote the area

$$\therefore A=\int_1^7 y^2dx=\left[\frac{y^3}{3}\right]^7$$

Substituting the given limits, we find

$$A=\frac{1}{3}(7^3-1^3)=\frac{342}{3}=114$$

To avoid mistakes it is advisable to write the indefinite integral in square brackets as shown, and afterwards to substitute and simplify.

It is instructive to compare the accurate result obtained by integration with the value found by using Simpson's Rule.

Ex. 18. To find the area of the curve $y=x^2$, between the values $x=1$ and $x=7$.

For values of $x=1$, 2, etc, calculate and tabulate values of y as follows:

x	1	2	3	4	5	6	7
y	1	4	9	16	25	36	49

By Simpson's Rule the area of the curve from 1 to 7, is

$$\frac{1}{3}\{1+49+4(4+16+36)+2(9+25)\}=\frac{342}{3}=114.$$

The area obtained is that of a parabola and therefore the result agrees with that obtained by integration. Simpson's parabolic rules give accurate results in such cases, even if only three equidistant ordinates are given Thus, in the preceding example, using the three ordinates 1, 16 and 49. Then, as the common distance is 3,

$$\text{area}=\frac{3(1+49+4\times 16)}{3}=\frac{342}{3}=114$$

In fact Simpson's Rule proceeds on the assumption that the curve is a parabola, and consequently the nearer any given case approaches to this form, the greater the accuracy obtained by the rule.

Ex 19 Find the volume of a log of timber 36 feet long, the areas of cross-sections at equal intervals of 6 feet being as follows:
8·20, 5 68, 4 04, 2 92, 2 16, 1 54, 1 02, sq ft respectively

I. **Simpson's First Rule**

Sum of end ordinates $=8{\cdot}20+1\ 02=9{\cdot}22$

,, ,, even ,, $=5{\cdot}68+2\ 92+1\ 54=10\ 14$

,, ,, odd ,, $=4\ 04+2\ 16=6\ 20.$

$$\text{Volume}=\tfrac{6}{3}(9\ 22+4\times 10\ 14+2\times 6\ 20)$$
$$=2\times 62\ 18=124\ 36 \text{ cub ft}$$

II **Simpson's Second Rule**

$$V=\frac{3\times 6}{8}\{8{\cdot}20+1{\cdot}02+2\times 2\ 92+3(5\ 68+4\ 04+2\ 16+1\ 54)\}$$
$$=\tfrac{9}{4}(9\ 22+5\ 84+3\times 13\ 42)$$
$$=124{\cdot}47 \text{ cub ft}$$

III. **Weddle's Rule.**

$$V=\frac{3\times 6}{10}\{8\ 20+4{\cdot}04+2{\cdot}92+2\ 16+1\ 02+5(5\ 68+2{\cdot}92+1{\cdot}54)\}$$
$$=\tfrac{18}{10}(18{\cdot}34+50{\cdot}70)$$
$$=\frac{9\times 69{\cdot}04}{5}=124{\cdot}272 \text{ cub. ft.}$$

IV. **Trapezoidal Rule.**

$$V=6\{\tfrac{1}{2}(8\ 20+1\ 02)+5\ 68+4\ 04+2\ 92+2{\cdot}16+1\ 54\}$$
$$=6\times 20{\cdot}95=125\ 7 \text{ cub. ft}$$

V **Mid-ordinate Rule.** Adding the 7 given values the sum is 25·56;

$$\therefore\ \text{Area}=\frac{25\ 56}{7}\times 36=131\ 45 \text{ cub. ft}$$

It is important to be able to use more than one method in calculating the area or volume of a given irregular figure, the results obtained by one method may be used as a check on the other.

Ex 20. Plot the curve $y=2{\cdot}45e^{0\ 4x}$ (1) where $e=2\ 718$ Find the average value of y from $x=0$ to $x=8$

When values 0, 1, 2, 3 are assumed for x corresponding values of y can be calculated. Thus, when $x=0$, from (1)

$$y=2\ 45e^0=2\ 45.$$

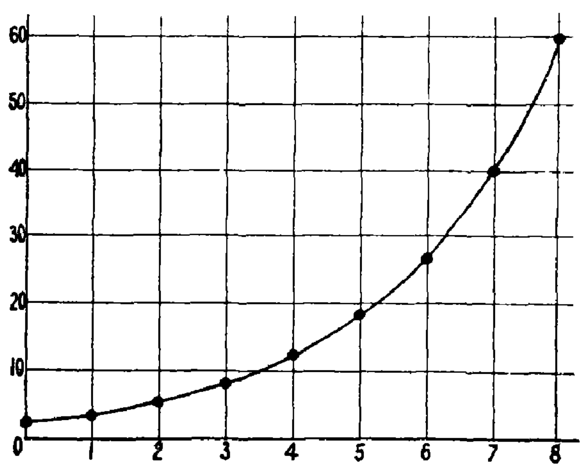

FIG 142 —Graph of $y=2\ 45e^{0\ 4x}$

When $\quad x=3,\ y=2\ 45e^{3\times 0\ 4}=2\ 45e^{1\ 2},$

$$\log y=\log 2\ 45+1{\cdot}2\log 2{\cdot}718=0\ 91036\ ,$$

$$\therefore\ y=8{\cdot}136$$

Values of x and corresponding values of y are given in the following table.

x	0	1	2	3	4	5	6	7	8
y	2·45	3·656	5·453	8·136	12·13	18·10	27·01	40·29	60·12

The curve is shown in Fig. 142.

The area OAB may be obtained by Simpson's Rule as follows:

Sum of end ordinates $= 2{\cdot}45 + 60{\cdot}12 = 62{\cdot}57$,

Sum of even ordinates $= 3{\cdot}656 + 8{\cdot}136 + 18{\cdot}10 + 40{\cdot}29 = 70{\cdot}182$,

Sum of odd ordinates $= 5{\cdot}453 + 12{\cdot}13 + 27{\cdot}01 = 44{\cdot}593$.

$$A = \tfrac{1}{3}(62{\cdot}57 + 4 \times 70{\cdot}182 + 2 \times 44{\cdot}593) = \frac{432{\cdot}478}{3} = 144{\cdot}16.$$

Also $\qquad$ area $=$ (average ordinate) $\times$ (length of base),

$$\text{average value} = \frac{144{\cdot}16}{8} = 18{\cdot}02.$$

The preceding result may be obtained more accurately by integration. Thus, if $y = Ae^{ax}$, $\dfrac{dy}{dx} = aAe^{ax}$.

Hence $$\int Ae^{ax}dx = \frac{1}{a}Ae^{ax} + C.$$

Ex. 21. If $y = 2{\cdot}45e^{0{\cdot}4x}$, find the average value of y from $x=0$ to $x=8$.

$$(\text{average value}) \times 8 = \int_0^8 2{\cdot}45e^{0{\cdot}4x} = \frac{2{\cdot}45}{0{\cdot}4}\Big[e^{0{\cdot}4x}\Big]_0^8$$

$$= 6{\cdot}125\,(e^{3{\cdot}2} - 1)$$

$$= 6{\cdot}125 \times 23{\cdot}54 = 144{\cdot}18;$$

$$\text{average value} = \frac{144{\cdot}18}{8} = 18{\cdot}02.$$

Some applications of integration.—Many of the rules and formulae used in mensuration are extremely difficult or impossible to obtain by elementary algebraical methods. There are very few, however, which do not yield to an elementary application of the calculus. The proofs of some of those which are of constant occurrence in practical work are given in the following pages, others may, if necessary, be obtained by similar methods.

Area of a circle.—Let θ denote the angle BCD (Fig. 143), then DCE, a small increase in the angle, will be denoted by $\delta\theta$, and the arc $DE = r\delta\theta$ Draw the chord DE. The area of the triangle $DCE = \frac{1}{2}DE \times CF$ where CF is drawn perpendicular to DE

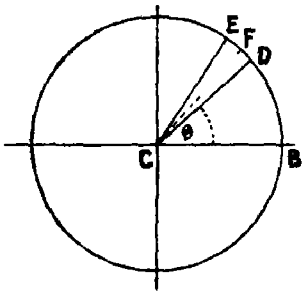

Fig. 143 —Area of a circle

When the angle becomes indefinitely small, the arc DE becomes equal to the chord DE, and CF becomes r

$$\text{area of triangle} = \tfrac{1}{2}rd\theta \times r$$
$$= \tfrac{1}{2}r^2 d\theta.$$

The sum of all such triangles, θ varying from 0 to 2π, will give the area of the circle

$$\text{area} = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \frac{r^2}{2}\int_0^{2\pi} d\theta\,,$$

$$\text{area} = \frac{r^2}{2}\left[\theta\right]_0^{2\pi} = \pi r^2.$$

Surface of a cone.—Let r denote the radius of the base, l the length of the slant side ON (Fig 144), then if AD, BC denote two plane sections perpendicular to the axis of the

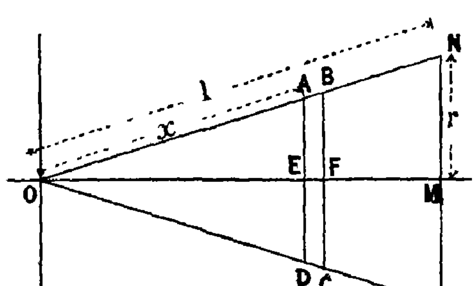

Fig 144 —Surface of a cone

cone which cut the cone in two circles shown by the lines AED, BFC respectively, when these planes lie close together, if y denotes the radius AE, the surface

of the slice $ABCD$ is $2\pi y \times EF$ approximately, and this approaches closer and closer to the actual value as the distance EF is made smaller. When the two points E and F are indefinitely near to each other, the points A and B form two consecutive points on the surface, and the expression for the area becomes $2\pi y dx$ .... (1)

It is now necessary to express y in terms of x. Let $OA = x$, then, from the similar triangles OEA and OMN,

$$x : y = l : r,$$

$$y = \frac{xr}{l}.$$

Substitute in (1),

$$\text{area of slice} = \frac{2\pi r x}{l} dx$$

The total surface from $x=0$ to $x=l$ may be denoted by S

$$S = \int_0^l \frac{2\pi r}{l} x dx = \frac{2\pi r}{l} \int_0^l x dx$$

$$= \frac{2\pi r}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{2\pi r l^2}{2l}$$

$$= \pi r l.$$

Hence, we obtain the rule :—**The curved surface of a cone is one-half the perimeter of the base multiplied by the slant height.**

When the total surface is required it is necessary to add the area of the base to this ;

$$\text{total surface} = \pi r l + \pi r^2$$

$$= \pi r (l + r)$$

Surface of a sphere.—Any plane section, such as BC, cuts the sphere in a circle ; let AD be any other section drawn parallel, and indefinitely near to BC and on the side of BC nearest to the centre of the sphere, then the radius of the circle is slightly larger than that of plane BC.

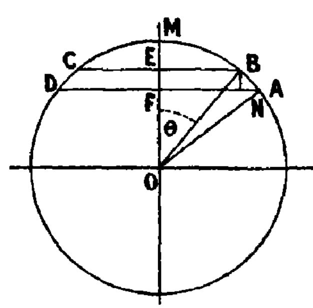

FIG 145.—Surface and volume of a sphere

Let x denote the distance ME, then, when the two sections are indefinitely near to each other, the distance FE will be denoted by dx.

The portion $ABCD$ is a flat circular plate of radius BE and thickness dx.

Join the points B and A to the centre O, then if the angle EOB be denoted by θ, the angle BOA will be represented by $d\theta$

$$\text{Area of slice } ABCD = 2\pi \times BE \times AB. \quad (1)$$

Now $\quad BE = r \sin\theta$ and $AB = r\,d\theta$

Substituting in (1),

$$\text{Area of } ABCD = 2\pi r \sin\theta \times r d\theta$$

The sum of all such slices from $\theta=0$ to $\theta=\frac{\pi}{2}$ will give the surface of the hemisphere and twice this sum will be the surface of the sphere,

$$\therefore \text{ surface of sphere} = 2\int_0^{\frac{\pi}{2}} 2\pi r^2 \sin\theta d\theta$$

$$= 4\pi r^2 \int_0^{\frac{\pi}{2}} \sin\theta d\theta = 4\pi r^2 \Big[-\cos\theta\Big]_0^{\frac{\pi}{2}}$$

$$= 4\pi r^2$$

Volume of a sphere.—Let AD and BC be two plane sections of the sphere when the two planes are indefinitely near to each other, or a distance dx apart The volume of the slice $ABCD$ is that of a flat circular plate of radius BE, and thickness dx

Join the centre O to points B and A Then, if θ denotes the angle EOB, the angle BOA, a slight increase to θ, will be denoted by $d\theta$,

$$\text{volume of } ABCD = \pi \times BE^2 \times dx.$$

Let r denote the radius of the sphere

Then $\quad BE = r\sin\theta,$

and $\quad BN$ or $dx = AB \sin BAN$

$$= r d\theta \sin\theta;$$

$$\therefore \text{ volume of } ABCD = \pi r^2 \sin^2\theta \times r \sin\theta d\theta$$

$$= \pi r^3 \sin^3\theta d\theta$$

The sum of all such slices from 0 to $\frac{\pi}{2}$ will give the volume of the hemisphere, and twice this sum is the volume of the sphere

$$\text{volume of sphere}=2\int_0^{\frac{\pi}{2}}\pi r^3\sin^3\theta d\theta=2\pi r^3\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta$$

$$=2\pi r^3\int_0^{\frac{\pi}{2}}\sin\theta(1-\cos^2\theta)d\theta=2\pi r^3\left[\int_0^{\frac{\pi}{2}}\sin\theta d\theta-\int_0^{\frac{\pi}{2}}\cos^2\theta\sin\theta d\theta\right]$$

$$=2\pi r^3\left[-\cos\theta-\left(-\frac{\cos^3\theta}{3}\right)\right]_0^{\frac{\pi}{2}}$$

$$=2\pi r^3\left[\frac{\cos^3\theta}{3}-\cos\theta\right]_0^{\frac{\pi}{2}}=2\pi r^3\times\frac{2}{3}=\frac{4}{3}\pi r^3;$$

or, let MO be taken to be the axis of x, and let x, y denote the co-ordinates of B Then

$$\text{volume of strip}=\pi y^2dx,$$

if r denote the radius and V the volume of the sphere,

$$\frac{V}{2}=\pi\int_0^r y^2dx=\pi\int_0^r(r^2-x^2)dx=\pi\left[r^2x-\frac{x^3}{3}\right]_0^r=\frac{2}{3}\pi r^3;$$

$$V=\frac{4}{3}\pi r^3$$

Volume of a cone —Let r denote the radius of the base, and h the length of the axis of the cone Any plane section parallel to the base will be a circle

Let AD and BC (Fig. 146) be two such sections; then when the distance between the sections is indefinitely small, or dx, the area of AD is nearly equal to that of BC

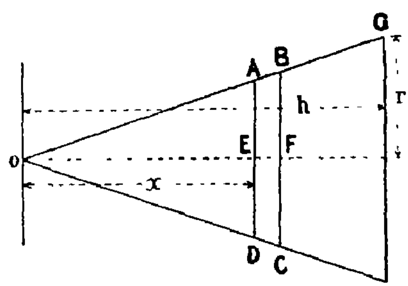

FIG 146 —Volume of a cone

Let $\qquad OE=x$ and radius $EA=y$

Volume of $ABCD=\pi y^2dx \qquad \ldots\ldots(1)$

The cone may be supposed to be divided into a large number of such small sections and the sum of all such will be the volume of the solid.

In Eq. (1) it is necessary to express y in terms of x.

Thus, from the similar triangles OHG and OEA, we find

$$x : y = h : r; \quad \therefore y = \frac{rx}{h}.$$

Substitute this value in (1);

$$\text{volume of } ABCD = \frac{\pi r^2 x^2}{h^2} dx.$$

If V denote the volume of the cone,

$$V = \int_0^h \frac{\pi r^2 x^2}{h^2} dx = \frac{\pi r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h,$$

$$V = \frac{\pi r^2 h}{3},$$

$\therefore$ **volume of cone is one-third the product of area of base and height**, or one-third the volume of a cylinder on the same base and the same height

Volume of a paraboloid.—It follows from the equation of a parabola $y^2 = 4ax$, that for each value of x, two values of y, equal in magnitude but opposite in sign, may be obtained Hence, the curve when plotted is symmetrical about the axis of x Further, as $x = 0$ gives $y = 0$, the vertex of the curve passes through the origin If the curve be assumed to rotate about the axis of x, it will generate a solid of revolution called a paraboloid of revolution. Two plane sections, such as AD and BC (Fig 147) will cut the solid in two circles whose centres are E, F respectively.

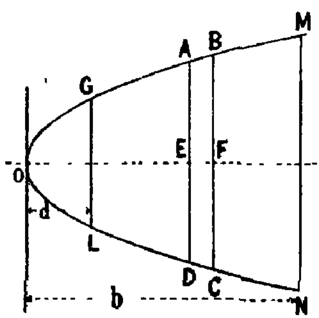

FIG 147 —Volume of a paraboloid

The volume of the portion $ABCD$ may be taken to be

$$\pi \times AE^2 \times EF = \pi y^2 \delta x,$$

the approximation becoming closer and closer to the actual value as the distance δx is diminished.

When δx is indefinitely small the volume of the slice $ABCD = \pi y^2 dx = 4\pi a x dx$, and the volume of the solid between the planes GL and MN at distances b and d from the origin respectively, is given by

$$V = \pi \int_d^b y^2 dx = 4\pi a \int_d^b x dx$$
$$= 4\pi a \left[\frac{x^2}{2}\right]_d^b = 4\pi a \left(\frac{b^2 - d^2}{2}\right)$$

If the volume be estimated from the origin, then $d=0$

$$V = \frac{4\pi a b^2}{2} = 2\pi a b^2$$
$$= \tfrac{1}{2}\pi c^2 b \text{ if } c^2 = 4ab$$

(c being the value of y when $x=a$). Therefore volume of segment of paraboloid of revolution is equal to one-half volume of cylinder on same base and same height

Ex 22 In the curve

$$y = cx^{\frac{1}{2}}, \qquad \text{(i)}$$

find c if $y=m$ when $x=b$. Let this curve rotate about the axis of x; find the volume V enclosed by the surface of revolution between the two section-planes at $x=a$, and $x=b$

Also find the numerical value of V when $m=6$, $b=4$, $a=2$, and $c=3$.

Substituting the given values of y and x in (i);

$$m = cb^{\frac{1}{2}};$$
$$c = mb^{-\frac{1}{2}} \qquad \text{(ii)}$$

It will be seen that as Eq (i) can be written in the form $y^2 = c^2 x$, it follows that the curve is one-half of a parabola, and therefore by revolution it will generate a paraboloid of revolution.

As in Fig. 147, the volume of a portion $ABCD$ is $\pi y^2 dx$;

$$V = \pi \int_a^b y^2 dx$$

Now express y in terms of x and the two constants of m and b, substitute the value of c from (ii) in (i), and we obtain

$$y^2 = \frac{m^2}{b} x,$$

$$\therefore V=\frac{m^2\pi}{b}\int_a^b x\,dx=\frac{\pi m^2}{b}\left[\frac{x^2}{2}\right]_a^b$$

$$=\frac{\pi m^2(b^2-a^2)}{2b}.$$

Substitute the given values of a, b, and m;

$$V=\frac{\pi\times 36(16-4)}{8}=54\pi$$

Prolate spheroid.—If an ellipse rotates about an axis passing through its major axis, it generates a solid of revolution called a prolate spheroid If the semi-axes of the ellipse MO and NO (Fig 148), are a and b respectively, the equation to the ellipse may be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots \quad (1)$$

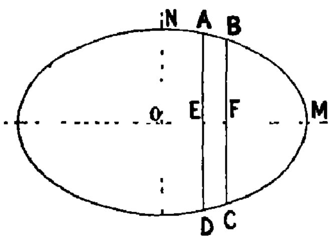

FIG 148 —Volume of a prolate spheroid

It will be noticed that when in Eq (1) $x=0$, $y=\pm b$; and $y=0$, $x=\pm b$ Hence, the centre of the ellipse is at the origin

Two plane sections AD and BC will cut the spheroid in two circles of radii AE and BF respectively. Let $AE=y$; then, if the distance EF be assumed to be indefinitely small, and denoted by dx,

$$\text{Volume of slice } ABCD=\pi y^2dx$$

If V denote the volume,

then
$$V=2\pi\int_0^a y^2dx$$

From (1)
$$y^2=\frac{b^2}{a^2}(a^2-x^2),$$

$$V=\frac{2\pi b^2}{a^2}\int_0^a (a^2-x^2)dx=\frac{2\pi b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a=\frac{4}{3}\pi ab^2.$$

It will be noticed that when $b=a$, the volume is that of a sphere of radius a

Oblate spheroid.—The volume generated by an ellipse rotating about its minor axis is called an *oblate spheroid*, and the volume may be obtained as in the preceding case.

Ex. 23 A curve whose equation is $4y^2=x^3$ is supposed to turn about the axis of x and trace out a surface of revolution Find the volume of the solid enclosed by the surface of revolution and the two circles traced out by the ordinates at $x=1$ and $x=4$ respectively

Denoting the volume enclosed by the surface of revolution, and the two circles traced out by the ordinates at $x=1$ and $x=4$ by V,

$$V=\pi\int_1^4 y^2dx=\frac{\pi}{4}\int_1^4 x^3dx\,,$$

$$V=\frac{\pi}{16}(256-1)=\frac{255\pi}{16}=50 \text{ cub. ft approx.}$$

EXERCISES XLII

Write down the values of

1 $\quad \int_1^4 x^2dx,\quad \int_{\frac18}^1 \frac{dx}{x}$

2 $\quad \int x^2dx,\quad \int(\cos bx)dx,\quad \int_{\frac14}^1 \frac{dx}{x},\ \text{and}\ \int_9^{12} x^2dx,$

where a and b are constants

3 Find $\int_a^b 2(c+x)dx$, when $a=10$, $b=20$, $c=4$

Give the values of

4 $\quad \int_a^b 3(c+nx^2)^2 2nx\,dx$, when $a=4$, $b=6$, $c=4$, and $n=2$.

(Hint, put $c+nx^2=z$, then $2nx\,dx=dz$)

Integrate the following

5 $\quad \int \cos ax\,dx$

6 $\quad \int \sec^2 ax\,dx$

7 $\quad \int \frac{1}{1+a^2x^2}dx$

8 $\quad \int e^{ax}dx$

9 $\quad \int A\cos(a+bx)dx.$

10 $\quad \int \frac{1}{1+(a+bx)^2}dx$

11 $\quad \int (p+qx)^2dx$

12 $\quad \int \frac{1}{\sqrt{1-(a+bx)^2}}$, where $a+bx<1$

Integrate with respect to x the following functions:

13. $ax^m dx,\ a + bx^n dx,\ \cos(a+bx)\,dx,\ \dfrac{dx}{x},\ \dfrac{dx}{a+bx}.$

14. $\dfrac{dx}{a^2+x^2}$ **15** $\dfrac{x\,dx}{a^2-x^2}.$

16 $\dfrac{dx}{x\sqrt{x^2-a^2}}.$ **17** $x^{\frac{p}{q}}\,dx.$

18. $\dfrac{x\,dx}{\sqrt{a^4-x^4}}$ **19** $\dfrac{x\,dx}{(x+1)(x+3)}$

20. $\dfrac{dx}{\cos^2\theta - \sin^2\theta}$ **21** $\dfrac{1+\cos\theta\,d\theta}{\theta+\sin\theta}$

22. $\dfrac{x\,dx}{(a^2-x^2)^{\frac{1}{2}}}.$ **23** $\dfrac{\tan^2 x\,dx}{4+\tan^2 x}$

Integrate the following:

24. $x^5dx,\ x^{\frac{1}{5}}dx,\ 2x^{\frac{1}{3}}dx,\ \sqrt[3]{x^2dx}.$

25 $(2x^2+3x+5)\,dx.$ **26** $(\sin x + \cos x)\,dx$

27. $(2\sin 4x\cos 2x)\,dx$ **28** $(\sin 2x\cos 4x)\,dx$

29. $(2\sin 4x\sin 2x)\,dx$ **30** $(2\cos 4x\cos 2x)\,dx$

31 $e^{av}dv$ **32** $av^{-1\cdot 37}dv$

33 $(at^2+bt+c)\,dt$ **34** $\dfrac{dx}{\sqrt{x^2+a^2}}$

35. $a^{m+x}dx$ **36** $\dfrac{dx}{x\sqrt{a^2+x^2}}$

37. $x^3(1+x^2)^{-\frac{1}{2}}\,dx$ **38.** $\dfrac{x\,dx}{x^4+a^4}$

39. $\dfrac{dx}{x^3(a+bx)}.$ **40** $\dfrac{x^3\,dx}{\sqrt{1-x^2}}$

41. $(\sin x)^2dx.$ **42** $\dfrac{x^2-1}{x^2-4}\,dx$ **43.** $2{\cdot}4^{2x}\,dx.$

44. $\dfrac{dx}{\cos x}.$ **45.** $\dfrac{dx}{\sin x}.$

46. There is a curve whose shape may be drawn from the following values of x and y.

x	0	1	2	3	4	5	6	7	8
y	0	1·25	5	11·25	20	31·25	45	61·25	80

Find the relation connecting x and y.

Assuming this curve to rotate about the axis of y, find the volume enclosed by the surface so traced and the end sections where $x=0$ and $x=8$

47 The shape of a curve may be obtained from the following values of x and y

Assuming this curve to rotate about the axis of x, find the volume of the solid between the values $x=0$ and $x=32$

x	0	3	5	7	9	12	16	19	21	23	26	30	32
y	15	12·9	12·35	12·36	12·6	13·6	15·55	16·34	16·6	15·6	13·18	8·7	5·7

CHAPTER XX

CENTRE OF GRAVITY MOMENT OF INERTIA

Moment of a force.—The **moment** of a force, about a given point, is the product of the force and the perpendicular let fall from the given point on the line representing the direction of the force.

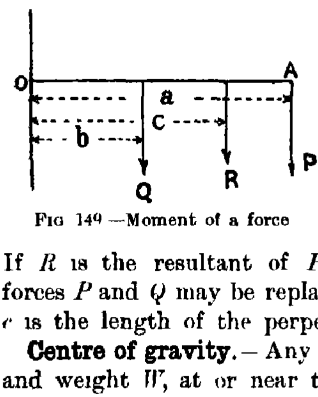

FIG 149 —Moment of a force

Thus, let P be a force (Fig 149) acting at A, and O the given point From O, draw OA perpendicular to the direction of P If a denotes the length of this perpendicular, then the moment of P is Pa Similarly, the moment of Q about O is Qb If R is the resultant of P and Q, i e $R=P+Q$, then the forces P and Q may be replaced by R, if $R \times c = Pa + Qb$, where c is the length of the perpendicular from O on R

Centre of gravity.— Any small portion of matter, of mass M and weight W, at or near the Earth's surface, is acted on by a force $W=Mg$ (where g is the acceleration due to gravity) As a body may be assumed to be an aggregate of small parts, and the forces due to these constitute a large number of parallel forces, the single force (or resultant) equal to their sum is called the **weight of the body**. The point in a body at which this single force may be assumed to act, whatever be the position of the body, is called the **centre of gravity** of the body The term is, for convenience, used to denote a centre of an area, a centre of figure, or even a linear centre Such a point is in many cases easily obtained. Thus, it would be the centre

of a circle, the point of intersection of the diagonals of a rectangle, etc

The centre of gravity of an irregular figure, especially when of comparatively small size, may be obtained by experimental methods. Thus, with a template, the exact shape of the figure may be cut out of a sheet of tin, cardboard, zinc, etc., and when such a template is freely suspended, the centre of gravity is in the vertical line passing through the point of support In this manner two vertical lines can be drawn, and the point of their intersection is the centre of gravity of the figure Another convenient method is to balance the figure on a knife edge and mark the line on it along which the figure balances, two such lines determine, as before, the position of the centre of gravity

There are comparatively few bodies which have a centre of gravity, what is usually meant is the **centre of mass,** or **centre of area.**

Ex 1 Find the centre of gravity of four bodies, weights 4, 2, 3, and 1 respectively, and arranged as in Fig 150, their distances from a point O being 2, 7, 11, and 13 units of length respectively

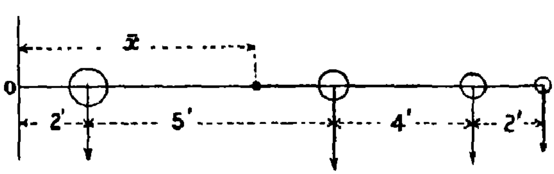

FIG 150 —Centre of gravity

Let the four weights in Fig 150 be assumed to be rigidly connected together by a weightless rod or wire To find the centre of gravity, or the point where a single force can be applied so that they remain in equilibrium, we may proceed as follows·

The four bodies shown give rise to four parallel forces, the sum of the moments of these four forces about any line, such as *oy*, must be equal to the *moment of* the resultant about the same line. Let $\bar{x}$ denote the distance of the resultant from *oy*.

Then, the sum of the moments will be

$$(4\times 2)+(2\times 7)+(3\times 11)+(1\times 13)=68.$$

The moment of the resultant is

$$(4+2+3+1)\bar{x};$$

$$10\bar{x}=68,$$

or

$$\bar{x}=6\cdot 8$$

Hence, the resultant acts at a point 6·8 from oy. If a single upward force equal to 10 were applied at this point, the system would remain in equilibrium

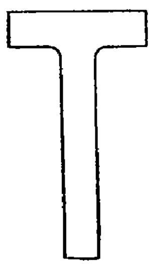

Fig 151 —Centre of area of a T-section

Ex. 2 Find the centre of area of the T-section (Fig 151)

The position of the centre of area may be obtained by taking moments about the upper edge Let x denote the distance of the centre of area from the upper edge

The areas of the two rectangles are $2'' \times \frac{1}{2}''$, and $4'' \times \frac{1}{2}''$, or 1 and 2 sq in respectively

Hence, $\bar{x} \times 3 = 1 \times \frac{1}{4} + 2 \times 2\cdot 5;$

$$x = \frac{5\cdot 25}{3} = 1\cdot 75$$

Ex 3. Find the centre of area of a section of a cast iron girder of the following dimensions flanges, $3'' \times 1''$ and $9'' \times 1''$; depth of girder, 12″; web, 1″ thickness.

To find the position of the centre of area, we divide the area (Fig 152) into three rectangles—those made by the two flanges and by the web W.

The areas of the flanges are 3×1 and 9×1, and that of the web is 10×1 sq in.

Hence, if $\bar{x}$ denotes the distance of the centre of area from the base AB,

Then

$$\bar{x}(3+9+10) = 9 \times 0\cdot 5 + 10 \times 6 + 3 \times 11\cdot 5;$$

$$\therefore 22\bar{x} = 99, \text{ or } \bar{x} = 4\cdot 5$$

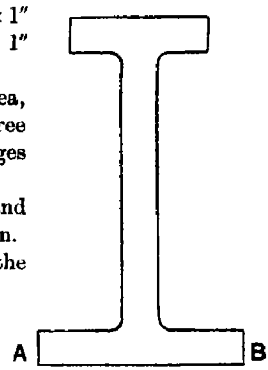

Fig 152 —Section of girder

Hence, G, the centre of area of the given figure, is at a distance $4\frac{1}{2}$ inches from the base AB.

If $ABCD$ (Fig 153) represents an irregular figure of uniform thickness, then the weights of the small strips into which the body may be assumed to be divided may be denoted by w_1,

w_2, w_3 .. and the distances of their centres from E by x_1, x_2, x_3 . Then, if $\bar{x}$ is the distance of the centre of gravity from E, and W the total weight,

$$\bar{x}=\frac{w_1x_1+w_2x_2+w_3x_3+\ldots}{w_1+w_2+w_3+\ldots}=\frac{\Sigma(wx)}{\Sigma w}$$

$$=\frac{\Sigma(wx)}{W}\quad\ldots\quad\ldots\ldots\quad\ldots\qquad.\qquad(\text{i})$$

The preceding equation will determine the position of the centre when the given body is symmetrical about a line such as EF When this is not the case, two calculations which are expressed by $\bar{x}=\frac{\Sigma wx}{W}$, $\bar{y}=\frac{\Sigma wy}{W}$ must be made.

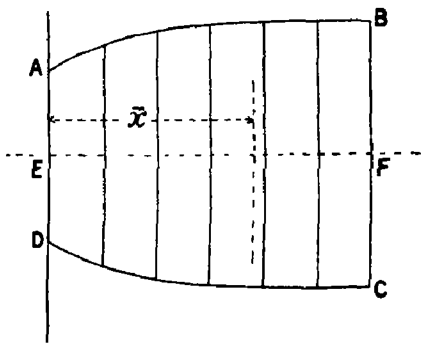

FIG 153 – Centre of area of an irregular figure

In the general case, the co-ordinates of the centre of gravity are obtained from

$$\bar{x}=\frac{\Sigma(wx)}{\Sigma w},\ \bar{y}=\frac{\Sigma(wy)}{\Sigma w},\ \bar{z}=\frac{\Sigma(wz)}{\Sigma w}\ .\ .\qquad .\ .\ .\ (\text{ii})$$

It is frequently convenient to apply the term centre of gravity to bodies which have no weight, such as geometrical figures, lines and planes In such cases we mean the point which would be the centre of gravity if the body was of uniform density, or its weight was proportional to its length, area or volume To obtain the weight of a body from its area of cross-section and length, it would be necessary to introduce common factors in Eq (ii), these could then be

cancelled, leaving simply volumes, areas, or lengths instead of weights

Application of Integration.—The centre of gravity of a surface in which the boundary consists of a curved line may be obtained approximately by Eq (ii) Strictly, however, the sub-divisions should be made indefinitely small, and the problem is therefore one requiring the integral calculus When the process of integration can be applied, it affords the most rapid and also the most accurate method of obtaining the centre of gravity Thus, if $\bar{x}$, $\bar{y}$, $\bar{z}$ have the same meanings as before,

then $\bar{x} = \dfrac{\int wxdx}{\int wdx}$, or $\bar{x} = \dfrac{\int mxdx}{\int mdx}$, where m denotes unit mass

Similar expressions hold for y and z Expressed in words, **the integral of the moments about a line of the small portions of mass into which a given body may be assumed to be divided, must be divided by the integral of the sum in order to obtain the distance of the centre of gravity from that line.**

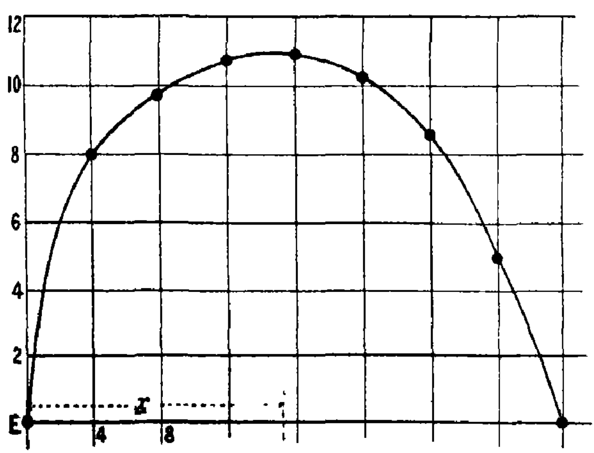

FIG 154 —Centre of area

Ex. 4 The half ordinates in feet of a symmetrical area (Fig 154) are 0, 8 0, 9·6, 10 8, 11 0, 10 2, 8·6, 5 0, 0 , find the area and the position of the centre of gravity, the common interval being 4 ft.

If the figure was drawn to scale, the ordinates shown by dotted lines could be measured, and each being multiplied by 4 the result would approximately denote the area of each strip If each area so obtained was multiplied by its distance from some point such as E, then by adding all the products together and dividing by the total area, the position of the centre of area is determined We may tabulate as before

(1) Ordinate,	0	8 0	9 6	10 8	11 0	10 2	8·6	5 0	0
(2) Simpson's Multiplier,	1	4	2	4	2	4	2	4	1
(3) Product,	0	32	19·2	43 2	22·0	40 8	17 2	20 0	0
(4) $\frac{\text{Moment}}{4}$,	0	32 0	38 4	129 6	88 0	204·0	103 2	140	0

The sum of the numbers in row (3) amounts to 194 4;

$$\text{area} = \tfrac{4}{3} \times 194\ 4 = 259\ 2$$

To obtain the centre of area, or centre of gravity, each product in row 3 is multiplied by its distance from E (Fig 154). Thus, $32\ 0 \times 1 \times 4$, $19\ 2 \times 2 \times 4$, $43\ 2 \times 3 \times 4$, etc

As the multiplier 4 occurs in each product, it is best to obtain the numbers as shown in row 3, find the sum of the numbers $= 735\ 2$, and finally multiply the sum by 4

Let $\bar{x}$ denote the distance of the centre of gravity from E, then

$$\bar{x} = \frac{735\ 2 \times 4}{194\ 9} = 15{\cdot}13$$

the distance of the centre of gravity from E is 15 13 ft

Guldinus's Theorems.—Suppose an area BG (Fig. 155) is connected by means of a thin bar GD to an axis OO in the plane of the area

If the area be made to revolve about the axis, it will generate a ring, the cross-section of which will be the area BG Let A denote the area of BG, and V denote the volume of the ring If a denotes an exceedingly small area at a distance,

y, from the axis, then in one revolution the volume generated is $2\pi ay$;

$$\therefore V=\Sigma(2\pi ay)=2\pi\Sigma(ay).$$

If $\bar{r}$ denote the distance of the centre of area from the axis, then

$$\Sigma(ay)=\bar{r}A, \text{ also } \Sigma a=A;$$

$$V=2\pi\bar{r}A \quad \ldots \quad \text{(i)}$$

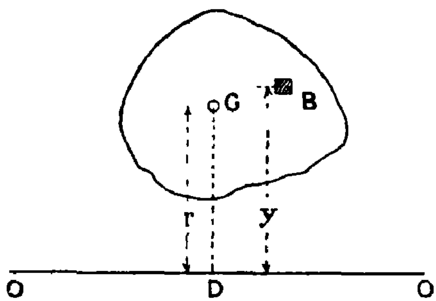

FIG 155 —To illustrate Guldinus's Theorem

This result may be expressed in words as follows

The volume generated by the revolution of a plane figure about any external axis in its plane is equal to the product of the area and the distance moved through by the centre of gravity of the area.

Thus, the volume traced out by an irregular figure can be obtained when the area and the position of the centre of area are known

Surface.—Let BG (Fig 155) denote a closed curve, then the revolution of the curve about the axis OO will generate a surface. A very short length of the curve, which may be denoted by δs, at a distance x from the axis, will generate a strip of area $2\pi x\delta s$, and if S denotes the whole surface generated, then

$$S=2\pi\Sigma(x\delta s)$$

If $\bar{r}$ denotes the distance of the centre of gravity and s the total length of the curve, then $\Sigma(x\delta s)=\bar{r}\Sigma\delta s=\bar{r}s$.

$\therefore$ surface generated $=2\pi\bar{r}s$, or in words, **the surface traced out by the revolution of a curve about an axis in its own plane is equal to the product of the perimeter of the curve into the distance moved through by the centre of gravity of the curve**

Conversely, when the length of a curve and the surface generated by the curve are known, the position of the centre of area or centre of gravity can be obtained (see p 221)

If a rectangle $ABCD$ (Fig. 156) revolves about one of its sides, as AB, it will trace out a cylinder, radius AD, and length AB When one side, as CE, is a curved line, the volume traced out by the figure may be obtained to any required degree of accuracy by using any of the approximate rules, Simpson's Mid-ordinate, etc

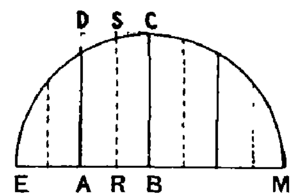

FIG 156

The volume, traced out by the figure ECM (Fig 156), may be found by dividing the figure into a number of parts, then, denoting the common distance AB by δx and the successive radii by y_1, y_2, etc, the volume traced out

$$=\delta x\{\pi y_1^2+\pi y_2^2+\quad\}$$

Ex 5. Find the volume traced out by the semicircle in Fig 156

As shown in Fig 156, the given figure is divided into four equal parts, the mid-ordinates being 1·3, 1 9, 1·9 and 1·3, the common distance 1,

$$\text{volume}=\pi\times 2(1\cdot3^2+1\ 9^2)=10\cdot6\pi$$
$$=33\cdot31$$

Ex 6 A circle $1\frac{3}{4}$ inches radius rotates about an axis 7 inches from the centre of the circle Find the surface and volume generated.

$$\text{Length of curve}=2\pi\times 1\tfrac{3}{4},\ \text{also}\ \bar{x}=7\,;$$
$$\text{surface generated}=2\pi\times 1\tfrac{3}{4}\times 2\pi\times 7=49\pi^2$$
$$=484\ \text{sq in}$$
$$A=\pi\times(1\tfrac{3}{4})^2\ \text{sq in.}\,;$$
$$\text{volume}=2\pi x A=2\pi\times 7\times\pi\times(1\tfrac{3}{4})^2$$
$$=14\pi^2(1\cdot75)^2=423\ 5\ \text{cub in}$$

Ex 7 There is a curve whose shape may be drawn from the following values of x and y

x in feet,	3	3 5	4 2	4 8
y in inches,	10 1	12 2	13 1	11·9

Imagine this curve to rotate about the axis of x, describing a surface of revolution

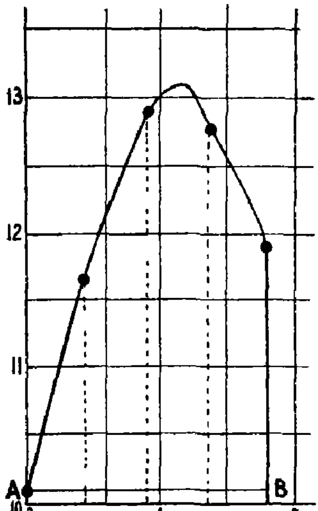

FIG 157 —Volume of an irregular figure

What is the volume enclosed by this surface and the two end sections where $x=3$ and $x=4\ 8$?

Plotting the given values of x and y, a curve, as in Fig 157, may be obtained

The base AB is $4{\cdot}8-3\ 0=1{\cdot}8$ Dividing this distance into 3 equal parts, the common distance is 0 6 ft., the mid-ordinates are 11 6, 12 9, and 12 7;

$$\therefore \text{ vol.}=0\ 6\times 12 \times \pi(11\ 6^2+12{\cdot}9^2+12{\cdot}7^2)$$
$$=3328\ 27\pi \text{ cub in}$$
$$=10456 \text{ cub in}$$
$$=6\ 05 \text{ cub ft}$$

Semicircle.—Let $EBDF$ (Fig 158) be a semicircle of radius r, and B and A two points on the circumference Through A and B draw two planes AD and BC parallel to the base EF, and at a distance apart represented by AN When the points A and B are indefinitely near to each other, the distance between the planes may be denoted by dx Join B to O Then, if θ denotes the angle BOE, the small increase to the angle BOE, shown by the small angle AOB, will be represented by $d\theta$ The area of the slice $ABCD=BCdx$, but

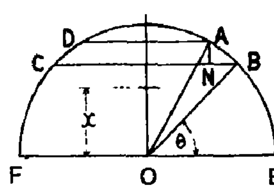

FIG 158- Centre of area of a semicircle

$$BC=2r\cos BOE=2r\cos\theta,$$

and $$dx=AB\cos\theta=r\cos\theta d\theta\ ,$$

$$\therefore \text{ area of slice } ABCD=2r^2\cos^2\theta d\theta,$$

$$\text{moment about } FE=2r^2\cos^2\theta d\theta\times x,$$

and $$x=r\cos AOE=r\sin\theta\ ,$$

$$\text{moment}=2r^3\cos^2\theta\sin\theta d\theta.$$

$$\text{Area of semicircle} = \frac{\pi r^2}{2};$$

$$\therefore \bar{x} \times \frac{\pi r^2}{2} = \int_0^{\frac{\pi}{2}} 2r^3 \cos^2\theta \sin\theta d\theta,$$

$$\bar{r} \times \frac{\pi r^2}{2} = 2r^3 \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta d\theta$$

$$= 2r^3 \left[-\frac{\cos^3\theta}{3} \right]_0^{\frac{\pi}{2}} = \frac{2r^3}{3};$$

$$\bar{r} = \frac{2r^3 \times 2}{3 \times r^2 \times \pi} = \frac{4r}{3\pi} = 0\ 4244r \text{ approx} \qquad (1)$$

Hemisphere.—Using the notation and diagram of the preceding case, the sections made by the two planes AD and BC will be circles of diameters AD and BC respectively The areas of the two circles may be taken to be the same when the distance between the planes is indefinitely small The volume of the portion $ABCD$ will be that of a flat circular disc of radius BE and thickness dx

$$\text{volume of } ABCD = \pi BE^2 \times dx,$$

$$BE = r\cos\theta \text{ and } dx = AB\cos\theta = r\cos\theta d\theta,$$

$$\text{mass of } ABCD = m\pi r^3 \cos^3\theta d\theta,$$

$$\text{moment of mass about base} = m\pi r^3 \cos^3\theta d\theta \times OE,$$

and $$OE = r\cos AOE = r\sin\theta,$$

$$\text{moment} = m\pi r^4 \cos^3\theta \sin\theta d\theta$$

Also, $$\text{mass of hemisphere} = \tfrac{2}{3} m\pi r^3,$$

$$\bar{x} \times \tfrac{2}{3} m\pi r^3 = \int_0^{\frac{\pi}{2}} m\pi r^4 \cos^3\theta \sin\theta d\theta,$$

or $$\bar{x} \times \tfrac{2}{3} m\pi r^3 = m\pi r^4 \int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta d\theta$$

$$= m\pi r^4 \left[-\frac{\cos^4\theta}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{m\pi r^4}{4},$$

$$\bar{x} = \frac{m\pi r^4 \times 3}{m \times 4 \times 2\pi r^3} = \tfrac{3}{8} r,$$

or, the centre of gravity is $\frac{3}{8}$ths of the radius, measured from the base of the hemisphere

Centre of gravity of a right cone.—Let the axis of the cone be horizontal and coincide with the axis of x as in Fig. 159. Let $\bar{x}$ denote the distance of the centre of gravity from an axis passing through the vertex O and parallel to the base.

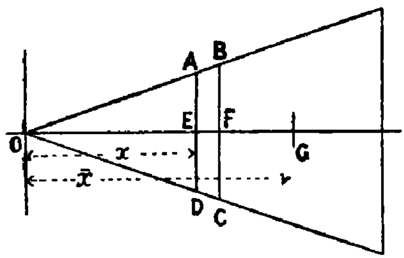

FIG. 159.—Centre of gravity of a right cone.

The sections made by two planes AD and BC, parallel to the base of the cone, will be two circles having slightly different radii. The radius AE, when the distance between the planes is indefinitely small, may be taken to be the same as BF.

Thus, volume $ABCD = \frac{\pi r^2 x^2}{h^2}\,dx$,

where r denotes the radius of the base and h the length of the axis at the cone.

If m is the mass of unit volume, then moment about O is

$$\frac{m\pi r^2 x^2}{h^2}\,x\,dx = \frac{m\pi r^2 x^3}{h^2}\,dx.$$

If $\bar{x}$ is the distance of the centre of gravity, then the total mass of the cone multiplied by $\bar{x}$ is equal to the sum of all the indefinitely thin slices into which the body is assumed to be divided.

$$\frac{m\pi r^2 h}{3} \times \bar{x} = \frac{m\pi r^2}{h^2}\int_0^h x^3\,dx$$

$$= \frac{m\pi r^2}{h^2}\left[\frac{x^4}{4}\right]_0^h = \frac{m\pi h^2 r^2}{4},$$

$$\therefore \bar{x} = \tfrac{3}{4}h;$$

or, the centre of gravity is at a point $\frac{3}{4}$ the length of the axis measured from O.

Moment of inertia.—**When the mass of every element of a body is multiplied by the square of its distance from a given axis, the product is called the moment of inertia about that axis.**

Moment of inertia of a thin rod.—The moment of inertia of a thin rod AB (Fig. 160), of length l, about an axis passing through one end and perpendicular to its length, is obtained as follows

The moment of inertia of a small element dx at a distance x from the axis is mx^2dx, where m denotes the mass of unit volume Hence, the moment of inertia of the rod will be

$$\int_0^l mx^2dx.$$

Denoting this expression by I, we have

$$I=\left[\frac{mx^3}{3}\right]_0^l=\frac{ml^3}{3}$$

If M denotes the total mass of the rod, then, since

$$M=ml,$$

$$I=\frac{Ml^2}{3} \quad . \quad . \quad (\text{i})$$

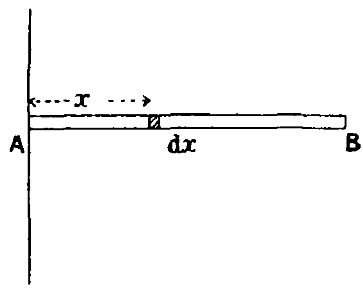

FIG 160—Moment of inertia of a rod

The value of I for an axis passing through the middle point of the rod, or through its centre of gravity, would be obtained in like manner, the limits of the integral being $\frac{l}{2}$ and $-\frac{l}{2}$,

or
$$I=\left[\frac{mx^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}=\frac{ml^3}{24}+\frac{ml^3}{24}=\tfrac{1}{12}ml^3$$

$$=\frac{Ml^2}{12}. \quad \ldots\ldots \quad . \quad . \ (\text{ii})$$

Ex. 1. Find the moment of inertia of a thin rod weighing 8 lbs. and 6 ft. long

(*a*) About an axis passing through one end and perpendicular to its length

(*b*) About an axis passing through its middle point and parallel to the preceding axis ($g=32$)

Here $\qquad M=\frac{8}{32}.$

(*a*) Substitute in (i), $\qquad I=\frac{8\times 6^2}{32\times 3}=3$

(*b*) $\qquad I=\frac{8}{32}\times\frac{36}{12}=\frac{3}{4}$

A convenient notation is to denote the moment of inertia of a given figure about an axis passing through the centre of area, or centre of gravity, by the symbol I_0, and about any parallel axis by the symbol I

Thus, the preceding result would be written as $I_0 = \frac{Ml^2}{12}$.

The moment of inertia of the rod about a parallel axis passing through one end may be deduced from the value of I_0, the proof of this theorem is very simple and may be left to the reader, i e the moment of inertia about any axis is equal to the moment of inertia about a parallel axis passing through the centre of gravity, together with the product of the mass and the square of the distance between the axes

$$I = I_0 + ml \times \left(\frac{l}{2}\right)^2$$

$$= \frac{1}{12} ml^3 + \frac{ml^3}{4} = \frac{ml^3}{3}$$

$$= \frac{Ml^2}{3}$$

Moment of inertia of a rectangle.—Let b denote the breadth, or width, of the rectangle and d its depth (Fig 161) The moment of inertia about an horizontal axis lying in the plane of the rectangle, passing through G the centre of gravity, may be obtained by assuming the figure to consist of an indefinite number of thin slices each of the thickness dx The moment of inertia of such a slice at a distance x from the axis (Fig 161) is $bdx \times x^2$, and the moment of inertia of the rectangle is the sum of all such slices,

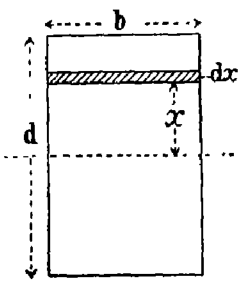

FIG 161 —Moment of inertia of a rectangle

$$I_0 = \int_{-\frac{d}{2}}^{\frac{d}{2}} bdx \times x^2 = b\int_{-\frac{d}{2}}^{\frac{d}{2}} x^2 dx$$

$$= \left[\frac{bx^3}{3}\right]_{-\frac{d}{2}}^{\frac{d}{2}} = \frac{bd^3}{12}.$$

Moment of inertia of a T-section.—The section, as in Fig. 151, consists of two rectangles The moment of inertia of

each rectangle about an axis passing through its centre of area can be obtained by substitution in the formula $I=\frac{1}{12}bd^3$.

Ex 2. Find the moment of inertia of the T-section (Fig 151) about an axis in the plane of the figure, and passing through the centre of area

The value of I for the upper rectangle is $\frac{1}{12}\times 2\times(\frac{1}{2})^3=\frac{1}{48}$, and for the lower rectangle $\frac{1}{12}\times\frac{1}{2}\times 4^3=\frac{8}{3}$

The moment of inertia of the whole section can now be obtained about any axes parallel to the preceding axes; one of the most useful axes is the line passing through the centre of area of the figure. Let I_0 denote the moment of inertia of the figure about the centre of area The distance between the axis of the upper rectangle and the line OO through the centre of area of the whole figure is $1\frac{1}{2}''=\frac{3}{2}''$. The corresponding distance for the lower flange is $\frac{3}{4}''$, also area of upper rectangle is $2''\times\frac{1}{2}''=1$ sq in, and the lower is $4\times\frac{1}{2}=2$ sq in

$$I_0=\frac{1}{48}+1\times(\tfrac{3}{2})^2+\tfrac{8}{3}+2\times(\tfrac{3}{4})^2$$
$$=\tfrac{291}{48}\text{ inch units}$$

In a similar manner the value of I for an axis passing through (say) the outer edge of the upper rectangle may be obtained.

Ex 3 Find the moment of inertia of the given cross-section (Fig 152) about an axis in the plane of the figure, and passing through G, the centre of area

The position of G has already been found to be at a distance of $4\frac{1}{2}$ inches from AB

The given section may be assumed to be divided into three rectangles, the value of I can be obtained and finally I_0.

For lower rectangle, $I=\dfrac{9\times 1^3}{12}$,

$$I_0=\frac{9\times 1^3}{12}+9\times 1\times 4^2=144.75\text{ inch units}$$

For upper rectangle, $I=\dfrac{3\times 1^3}{12}$;

$$I_0=\tfrac{1}{4}+3\times 1\times 7^2=147{\cdot}25$$

For web, $I=\dfrac{1\times 10^3}{12}$;

$$\therefore I_0=\frac{1000}{12}+10\times 1\times(1.5)^2=83{\cdot}33+22.5=105{\cdot}83;$$

$$I_0=144.75+147.25+105.83=397{\cdot}83\text{ inch units}$$

Moment of inertia of a thin disc.—The moment of inertia of a thin disc, of radius r, about an axis passing through the centre of the disc and perpendicular to its plane is obtained as follows

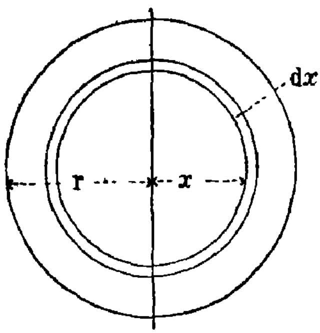

FIG 162 —Moment of inertia in a circle

The moment of inertia of an indefinitely thin annulus of thickness dx, and at a distance x from the axis (Fig 162), would be its area $2\pi x dx$ multiplied by the square of its distance from the given axis, or

$$2\pi x dx \times x^2 = 2\pi x^3 dx \quad . \qquad (i)$$

The value of I_0 will be the sum of an indefinite number of such annuli, or the sum of all such expressions as (i) from $x=0$ to $x=r$.

$$I_0 = \int_0^r 2\pi x^3 dx = 2\pi \left[\frac{x^4}{4}\right]_0^r$$

$$= \frac{\pi r^4}{2} \quad . \qquad (ii)$$

The moment of inertia of the area of a circle about any diameter is half the preceding result, or $\frac{\pi r^4}{4}$ This value is required when dealing with the bending of a beam of circular section, and may be readily obtained by taking, instead of annuli, strips or slices parallel to the diameter

Moment of inertia of a cylinder about its axis.—If r denotes the radius and l the length of the cylinder, m the mass of unit volume, then, as in the preceding case, the moment of inertia of an annulus of thickness dx, at a distance x from the axis, is the mass, $2\pi x dx l \times m$, multiplied by x^2 and $= 2\pi m l x^3 dx$;

$$I_0 = \int_0^r 2\pi m l x^3 dx = 2\pi m l \left[\frac{x^4}{4}\right]_0^r$$

$$= \frac{\pi m l r^4}{2} \quad . . \qquad . . .(ii)$$

If M denotes the total mass of the cylinder, then

$$M = m\pi r^2 l\,,$$

$$\therefore \text{ from (ii) } I_0 = \frac{Mr^2}{2}.$$

Moment of inertia of a hollow cylinder.—The moment of inertia of a hollow cylinder, external and internal radii R and r respectively, may be obtained by the preceding method, or inferred from (ii)

$$I_0 = \frac{\pi ml}{2}(R^4 - r^4),$$

but

$$M = \pi ml(R^2 - r^2)\,,$$

$$I_0 = \frac{M(R^2 + r^2)}{2}$$

It will be noticed that this result reduces to the preceding when $r = 0$

Radius of gyration.—It is often convenient to consider the total mass of a body as though it were concentrated at a point in a body The distance of this point from the axis is called the **radius of gyration**

Thus, the moment of inertia of a rod about one end is $\frac{1}{3}Ml^2$.

Let k denote the distance from the axis of a point such that the whole mass of the rod may be assumed to be collected, or to act, at the point, then,

$$I = \tfrac{1}{3}Ml^2 = Mk^2\,, \qquad k = \frac{l}{\sqrt{3}}$$

Similarly, as the **polar moment of inertia** of a circle of radius R is $\frac{MR^2}{2}$, the radius of gyration is given by

$$Mk^2 = \frac{MR^2}{2}\,; \qquad k = \frac{r}{\sqrt{2}}$$

In the case of a hollow circle or cylinder radii R and r respectively,

$$Mk^2 = \frac{M}{2}(R^2 + r^2)\,, \qquad k = \frac{1}{\sqrt{2}}\sqrt{R^2 + r^2}$$

Moment of inertia of a fly-wheel.—Usually a fly-wheel consists of a heavy rim connected by arms to its centre. In calculating the moment of inertia of such a wheel, only that of the rim is taken into account. If necessary, a small percentage of this may be added to the mass of the rim to allow for the arms and boss of the wheel. If R and r respectively denote the external and internal radii of the rim of the wheel, then the mean radius, or $\frac{1}{2}(R+r)$, is often taken as the radius of gyration.

It is easy to ascertain what amount of error is involved in this assumption when the magnitudes of R and r are given.

Ex. 4. For such a fly-wheel let $R=4$ and $r=3$.

Then $$\tfrac{1}{2}(R+r)=\tfrac{1}{2}(4+3)=3.5;$$

giving $$I_0=\frac{M(3.5)^2}{2}.$$

But $$k=\frac{1}{\sqrt{2}}\sqrt{4^2+3^2}=\frac{5\sqrt{2}}{2}=\frac{7.07}{2};$$

$$k=3.538,$$

$$I_0=\frac{M(3.538)^2}{2},$$

or an error of 2 per cent.

Ex. 5. Find the moment of inertia of a pulley, the cross-section being of the form shown in Fig. 163.

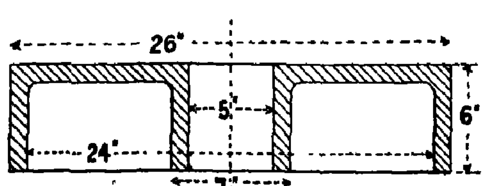

FIG. 163.—Moment of inertia of a pulley.

The moment of inertia of a disc or hollow cylinder can be obtained from tables, but a tabulated value cannot be obtained for the section of a wheel or pulley such as in Fig. 163, and the value of the moment of inertia must be obtained by calculation.

In such a case, we may assume the given section to be made up of three cylinders, the diameters of the outer one being 26 inches and 24 inches respectively and the length 6 inches The next is a cylinder of length or thickness 1 inch, and of 24 inches external and 7 inches internal diameter; and the dimensions of the inner are 7 inches and 5 inches diameter respectively, and 6 inches long The moment of inertia of the system is the sum of the moments of its separate parts

The value of I for a hollow cylinder about its geometrical axis is given by $\frac{M(R^2+r^2)}{2}$, where M denotes the mass, R and r the external and internal radii respectively.

The mass of a hollow cylinder is $\pi m(R^2-r^2)l$, where l denotes the length of the cylinder, and m the mass of unit volume of the material

$$\text{Mass of outer ring } A = \pi m\left\{\left(\frac{26}{2}\right)^2 - \left(\frac{24}{2}\right)^2\right\}6 = 150\pi m.$$

Similarly, the mass of the ring B is given by

$$\pi m\left\{12^2 - \left(\frac{7}{2}\right)^2\right\} = 131\ 75\pi m,$$

and mass of ring C is

$$\frac{\pi m}{4}(7^2-5^2)6 = 36\pi m$$

The moment of inertia of the whole will simply be the sum of the various rings into which the figure has been assumed to be divided

$$I = \pi m\left\{\left(150\times\frac{13^2+12^2}{2}\right)+\left(131\ 75\times\frac{12^2+\left(\frac{7}{2}\right)^2}{2}\right)\right.$$
$$\left.+\left(36\times\frac{\left(\frac{7}{2}\right)^2+\left(\frac{5}{2}\right)^2}{2}\right)\right\}$$

$$= \pi m(23475+10293+333)$$

$$= \pi m\times 34101$$

As the weight of 1 cub in. of cast iron is 0·26 lb ,

$$\therefore \text{ mass of 1 cub in} = \frac{0\cdot 26}{32\cdot 2}.$$

Hence $\quad I=\dfrac{\pi \times 0{\cdot}26 \times 34101}{32{\cdot}2}=865{\cdot}2$ lb inch units.

Usually the result is required in pound feet units, hence the preceding result must be divided by 12^2 or 144, giving $I=6{\cdot}08$ lb. ft. units.

Moment of inertia of a cylinder.—The moment of inertia of a cylinder, about an axis passing through its centre of gravity and perpendicular to its length, may be thus determined. Let r denote the radius and l the length of the cylinder. We may assume the cylinder to be composed of an indefinite number of thin discs each of thickness dx.

The moment of inertia of such a disc at a distance x from the axis (Fig. 164) is $\pi r^2 m \times x^2 dx$, where m denotes the mass of unit volume.

Mass of disc is $m \times \text{volume} = m\pi r^2 dx$

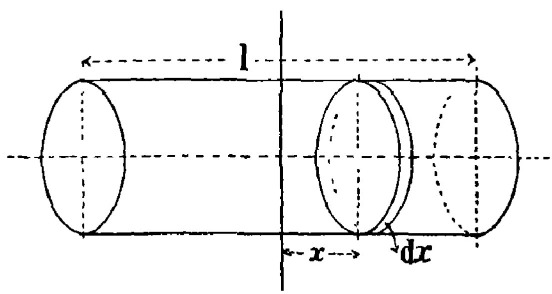

FIG. 164.—Moment of inertia of a cylinder.

Also the moment of inertia of the disc about its own diameter is $\dfrac{m\pi r^4}{4}dx$.

Hence, $$I_0=\pi m r^2\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx+\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{\pi r^4}{4}m\,dx$$

$$=\pi m r^2\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}+\frac{1}{4}\pi m r^4 l$$

$$=\frac{2\pi m r^2 l^3}{24}+\frac{1}{4}\pi m r^4 l \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Also, if M denotes the mass of the cylinder, then

$$M = \pi m r^2 l$$

Substitute in (i), and obtain

$$I_0 = M\left(\frac{r^2}{4} + \frac{l^2}{12}\right) \quad . \quad . \quad . \qquad \text{(ii)}$$

It follows at once from (ii) that if the radius of the cylinder is very small compared with its length, then the first term in (ii) may be neglected and the value of I_0 becomes $\frac{Ml^2}{12}$, as on p 431, for a thin rod

Similarly, if l is very small compared with r, we obtain $I = \frac{Mr^2}{4}$, the value of I for a thin disc

EXERCISES XLIII

In the following exercises the letters C G denote centre of gravity or centre of area, and the letter I denotes the moment of inertia about an axis passing through the centre of gravity and in the plane of the figure

1 The dimensions of a T-section, as in Fig 151, are as follows the upper flange is $2'' \times \frac{1}{2}''$ and the web $3'' \times \frac{1}{2}''$, find the distance of the C G from the extreme edge of the upper flange and the value of I about an axis passing through the C G in the plane of the figure

2 The dimensions of a rectangular strip of steel are. width 0 7", depth 0 1" If $E = 3\,6 \times 10^7$ and $M = 100$, find the value of r from the formula $r = \frac{EI}{M}$

3 The breadth or width of a rectangular beam is 2", its depth 3"; find the value of f from the formula $M = \frac{f}{y} I_0$, given $M = 8000$ and $y = 1$.

4. The flanges of a girder of the form shown in Fig. 152 are $4'' \times 1''$ and $6'' \times 1''$, and the web 1", the depth of the girder is 10" Find the distance of the C G from the outer edge of the larger flange and the value of I_0.

5. A form of rail section is given in Fig 165 Find the area of the cross-section, the position of its C G, the value of I_0, and the radius of gyration k Width of bottom $= 6\frac{1}{2}''$

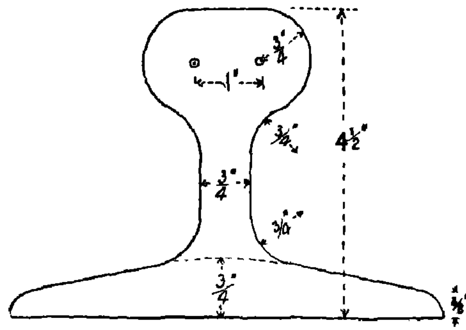

FIG. 165 —Rail section

6 ABC (Fig 166) is a segment of a parabola cut off by a chord AC normal to the axis, if b is the length of the chord and h its distance from the vertex B, show that its area is $\frac{2}{3}hb$ and its centre of gravity is $\frac{3}{5}h$ from B

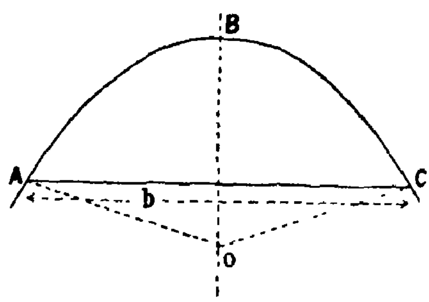

FIG. 166

7 If ABC (Fig 166) be assumed to be a circular sector centre O, radius r, and angle $AOB = \theta$, show that the distance of the centre of gravity from O is $\frac{2}{3}r\frac{\sin\theta}{\theta}$

CHAPTER XXI

INTEGRATION BY PARTIAL FRACTIONS INTEGRATION BY PARTS FOURIER'S SERIES FOURIER'S THEOREM.

Integration by partial fractions.—When it is required to integrate an expression of the form $\frac{7x-1}{1-5x+6x^2}$, in which the denominator can be resolved into the product of a series of linear or quadratic factors, as, in this case, $(1-3x)(1-2x)$, it is often the best way to break the fraction up into a series of partial fractions, p 6.

Thus
$$\frac{7x-1}{(1-3x)(1-2x)}=\frac{4}{1-3x}-\frac{5}{1-2x},$$
$$\int\frac{(7x-1)dx}{1-5x+6x^2}=4\int\frac{dx}{1-3x}-5\int\frac{dx}{1-2x}$$
$$=-\frac{4}{3}\int\frac{d(3x)}{3x-1}+\frac{5}{2}\int\frac{d(2x)}{2x-1}$$
$$=-\frac{4}{3}\log(3x-1)+\frac{5}{2}\log(2x-1)$$
$$=\log\frac{(2x-1)^{\frac{5}{2}}}{(3x-1)^{\frac{4}{3}}}$$

Ex 1 Integrate $\frac{x^2-7x+1}{x^3-6x^2+11x-6}$

Here the denominator is $(x-1)(x-2)(x-3)$
$$\int\frac{(x^2-7x+1)dx}{x^3-6x^2+11x-6}=\int\left\{-\frac{5}{2}\cdot\frac{1}{x-1}+\frac{9}{x-2}-\frac{11}{2}\cdot\frac{1}{x-3}\right\}dx$$
$$=-\frac{5}{2}\int\frac{dx}{x-1}+9\int\frac{dx}{x-2}-\frac{11}{2}\int\frac{dx}{x-3}$$
$$=-\frac{5}{2}\log(x-1)+9\log(x-2)-\frac{11}{2}\log(x-3).$$

When the denominator contains repeated factors, one or more of the constants may be determined, as in the preceding example, the remaining constants being obtained by differentiation. The method will be understood from the following example

Ex 2 Integrate $\dfrac{dx}{x^3-x^2-x+1}$

The factors of x^3-x^2-x+1 are $(x-1)^2(x+1)$

Let $$\frac{1}{x^3-x^2-x+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}. \qquad \text{(i)}$$

Notice that the two terms $\dfrac{A}{x-1}$ and $\dfrac{B}{(x-1)^2}$ occur for twice repeated roots Similarly, three terms would be used for three times repeated roots, etc

From (1)

$$1=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$
$$=A(x^2-1)+B(x+1)+C(x-1)^2 \qquad \text{(ii)}$$

Let $x=1$; $\qquad 1=2B$, or $B=\frac{1}{2}$

To determine the numerical values of A and C we may differentiate each side of equation (ii) for that equation obviously holds true for all values of x, and hence the differential coefficients of the two sides of the equation are equal

Differentiating (ii),

$$0=2Ax+B+2Cx-2C \qquad \text{(iii)}$$
$$=2Ax+\tfrac{1}{2}+2C(x-1)$$

Put $x=1$, $\qquad 2A=-\frac{1}{2}$; $\qquad A=-\frac{1}{4}$

Differentiating (iii),

$$2A+2C=0, \text{ or } 2C=\tfrac{1}{2},$$
$$C=\tfrac{1}{4}$$

Hence, substituting these values,

$$\frac{1}{x^3-x^2-x+1}=-\frac{1}{4(x-1)}+\frac{1}{2(x-1)^2}+\frac{1}{4(x+1)};$$

$$\int\frac{dx}{x^3-x^2-x+1}=\frac{1}{4}\int\frac{dx}{(x-1)}+\frac{1}{2}\int\frac{dx}{(x-1)^2}+\frac{1}{4}\int\frac{dx}{(x+1)}$$

$$=-\frac{1}{4}\log(x-1)-\frac{1}{2}\frac{1}{(x-1)}+\frac{1}{4}\log(x+1)$$

Integration by parts.—The differential of the product of u and v, where u and v are functions of x, has been obtained on p 320:

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}.$$

Hence, integrating,

$$uv=\int u\frac{dv}{dx}\,dx+\int v\frac{du}{dx}\,dx, \quad . \qquad \text{(iv)}$$

or

$$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$$

Thus, in the case of the product of two functions, in which the integral is not easily obtainable, it is possible by using Eq (iv) to express the integral in a form more easily dealt with than was the original expression, and thus, by successive steps, to reduce the unknown integrals to known forms

It is very important that the rule and its various applications should be clearly made out, and it is therefore advisable to commence with a few simple expressions which may easily be verified

Ex. 3 Let $u=3x^2$ and $v=4x^3$

Also $\frac{du}{dx}=6x$ and $\frac{dv}{dx}=12x^2$,

$$\int 3x^2\times 12x^2dx=3x^2\times 4x^3-\int 4x^3\times 6x\,dx\,;$$

or

$$\int 36x^4dx=12x^5-\int 24x^4dx\,,$$

$$\frac{36x^5}{5}=12x^5-\frac{24x^5}{5}=\frac{36x^5}{5}$$

Ex 4 Integrate $x^n\log x\,dx$

Let $u=\log x$ and $dv=x^n dx$;

$$\frac{du}{dx}=\frac{1}{x} \text{ and } v=\frac{x^{n+1}}{n+1}\,;$$

$$\int x^n\log x\,dx=\log x\,\frac{x^{n+1}}{n+1}-\int\frac{x^n}{n+1}dx$$

$$=\frac{x^{n+1}}{n+1}\left(\log x-\frac{1}{n+1}\right)$$

Ex. 5. Integrate $e^x \sin x dx$

Let $$u = \sin x \text{ and } dv = e^x dx,$$

$$\frac{du}{dx} = \cos x, \qquad v = e^x.$$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx \quad \ldots(\text{i})$$

Again $$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx, \quad (\text{ii})$$

by repeating the operation;

$$\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2},$$

by subtracting Eq (ii) from Eq. (i) and rearranging the terms

It will be noticed that it is often possible to obtain a solution by repeated integration by parts Especially is this the case when one of the factors is of the form x^n, in which case the application is made in such a way as to reduce the index each time, or, in other words, is denoted by u in the formula The method indicated will now be applied to obtain what are known as **reduction formulae**.

One of the most important formulae of reduction is that of $\sin^m\theta \cos^n\theta d\theta$, the integral being made to depend on another, in which the indices are reduced by two, and thus, by successive applications, the complete integral is obtained.

Since $$\int \sin^m\theta \cos^n\theta d\theta = \int \cos^{n-1}\theta \sin^m\theta d(\sin\theta), \quad \ldots(\text{i})$$

we may, in the formula for integration by parts, assume

$$u = \cos^{n-1}\theta, \qquad v = \frac{\sin^{m+1}\theta}{m+1},$$

$$\int \sin^m\theta \cos^n\theta d\theta = \frac{\cos^{n-1}\theta \sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2}\theta \cos^{n-2}\theta d\theta$$

Also $$\sin^{m+2}\theta = \sin^m\theta \times \sin^2\theta = \sin^m\theta(1-\cos^2\theta)$$

Substituting, we have

$$\int \sin^m\theta \cos^n\theta d\theta = \frac{\cos^{n-1}\theta \sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^m\theta(\cos^{n-2}\theta - \cos^n\theta)d\theta$$

$$= \frac{\cos^{n-1}\theta \sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^m\theta \cos^{n-2}d\theta - \frac{n-1}{m+1}\int \sin^m\theta \cos^n\theta d\theta;$$

transposing the last term and multiplying both sides by $\frac{m+1}{m+n}$, we obtain

$$\int \sin^m\theta \cos^n\theta d\theta = \frac{\cos^{n-1}\theta \sin^{m+1}\theta}{m+n} + \frac{n-1}{m+n}\int \sin^m\theta \cos^{n-2}\theta d\theta \quad . \text{(ii)}$$

It will be seen from (ii) that after integration by parts the integral is made to depend on another, in which the index of $\cos\theta$ is reduced by two In a similar manner, the integral (i) could be made to depend on another in which the index of $\sin\theta$ would be reduced by two Hence, by successive applications, the integral of $\int \sin^m\theta \cos^n\theta d\theta$ can always be reduced to that of $\int \sin\theta d\theta$, $\int \sin\theta \cos\theta d\theta$ or $\int \cos\theta d\theta$ when the indices are integers.

A very important case occurs when a definite integral, the limits being 0 and $\frac{\pi}{2}$, is required (m and n being integers).

Then $\displaystyle\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta d\theta = \frac{(m-1)(m-3) \ldots (n-1)(n-3)}{(m+n)(m+n-2) \ldots 4 \cdot 2} \times \phi$,

the quantity ϕ is unity, except when m and n are both even integers, in which case its value is $\frac{\pi}{2}$

Ex 6 Let $m=6$ and $n=4$ Here m and n are both even;

$$\int_0^{\frac{\pi}{2}} \sin^6\theta \cos^4\theta d\theta = \frac{5}{10}\cdot\frac{3}{8}\cdot\frac{1\times 3}{6}\cdot\frac{1}{4 \cdot 2} \times \frac{\pi}{2} = \frac{3\pi}{512}$$

Ex 7 Let $m=6$ and $n=5$,

$$\int_0^{\frac{\pi}{2}} \sin^6\theta \cos^5\theta d\theta = \frac{5}{11}\cdot\frac{3}{9}\cdot\frac{1\times 4}{7}\cdot\frac{2}{5}\cdot\frac{1}{3} = \frac{8}{693}$$

Ex. 8. $$\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta d\theta = \frac{1\times 3\times 1}{6\times 4\times 2} \times \frac{\pi}{2} = \frac{\pi}{32}$$

Ex. 9

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^3\theta d\theta = \frac{4 \cdot 2}{8}\cdot\frac{2}{6}\cdot\frac{2}{4}\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta d\theta = \frac{4}{8}\cdot\frac{2}{6}\cdot\frac{2}{4}\cdot\frac{1}{2}\Big[\sin^2\theta\Big]_0^{\frac{\pi}{2}} = \frac{1}{24}.$$

$\sin^n\theta d\theta$ and $\cos^n\theta d\theta$.—These examples may be taken to be special cases of the general formulae, but they are very important, especially in the case of definite integrals, and may be obtained independently as follows

Integrating by parts, we can connect

$$\int \sin^n\theta d\theta \text{ with } \int \sin^{n-2}\theta d\theta,$$

$$\therefore \int \sin^n\theta d\theta = -\frac{\cos\theta \sin^{n-1}\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta d\theta, \quad \text{(i)}$$

and
$$\int \cos^n\theta d\theta = \frac{\sin\theta \cos^{n-1}\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta d\theta \quad \text{(ii)}$$

From (i) we obtain, by successive applications,

$$\int \sin^n\theta d\theta$$
$$= -\frac{\cos\theta}{n}\left(\sin^{n-1}\theta + \frac{n-1}{n-2}\sin^{n-3}\theta + \frac{(n-1)(n-3)}{(n-2)(n-4)}\sin^{n-5}\theta + \text{etc}\right) + A,$$

when n is even, the last term in the bracket is

$$+\frac{(n-1)(n-3)}{(n-2)(n-4)} \cdots \frac{3}{2}\sin\theta, \text{ and } A = \frac{(n-1)(n-3)}{n(n-2)} \cdots \frac{3}{4}\frac{1}{2}\theta$$

When n is odd, the last term in the bracket is

$$+\frac{(n-1)(n-3)}{(n-2)(n-4)} \cdots \frac{2}{1}, \text{ and } A = 0$$

From (ii) we obtain, by successive applications,

$$\int \cos^n\theta d\theta$$
$$= \frac{\sin\theta}{n}\left(\cos^{n-1}\theta + \frac{n-1}{n-2}\cos^{n-3}\theta + \frac{(n-1)(n-3)}{(n-2)(n-4)}\cos^{n-5}\theta + \text{etc}\right) + A;$$

when n is even, the last term in the bracket is

$$+\frac{(n-1)(n-3)\ldots 3}{(n-2)(n-4)\ldots 2}\cos\theta, \text{ and } A = \frac{(n-1)(n-3)}{n(n-2)} \cdots \frac{3}{4}\frac{1}{2}\theta$$

When n is odd, the last term in the bracket is

$$+\frac{(n-1)(n-3)\ldots 2}{(n-2)(n-4)\ldots 1}, \text{ and } A = 0.$$

One of the most important applications of the integration of $\sin^n\theta d\theta$ is the definite integral between the limits 0 and $\frac{\pi}{2}$,

from (1),

$$\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta = -\frac{\cos\theta\sin^{n-1}\theta}{n} + \frac{n-1}{n}\int\sin^{n-2}\theta d\theta$$

When n is an integer, not less than 2, the first term becomes zero for both the limits $\theta=0$, $\theta=\frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta$$

$$= \frac{(n-1)(n-3)}{n(n-2)}\int_0^{\frac{\pi}{2}}\sin^{n-4}\theta d\theta, \text{ etc.}$$

This becomes, when n is even,

$$\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right) \quad \frac{5}{6}\ \frac{3}{4}\ \frac{1}{2}\int_0^{\frac{\pi}{2}}1 d\theta,$$

$$\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta = \frac{(n-1)(n-3)}{n(n-2)} \quad \frac{3\,.\,1}{4\ \ 2}\times\frac{\pi}{2} \text{ (n an even integer),}$$

and $$\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta = \frac{(n-1)(n-3)\,.\,.\,4\ \ 2}{n(n-2)\quad\ 5\ \ 3}\times 1 \text{ (n an odd integer)}$$

Ex 10 Let $n=4$,

$$\int_0^{\frac{\pi}{2}}\sin^4\theta\, d\theta = \frac{3\ \ 1}{4\ \ 2}\times\frac{\pi}{2} = \frac{3\pi}{16}$$

Ex 11 $$\int_0^{\frac{\pi}{2}}\sin^8\theta\, d\theta = \frac{7\,.\,5\ \ 3}{8\,.\,6\ \ 4\ \ 2}\times\frac{\pi}{2} = \frac{35\pi}{256}.$$

Ex 12 $$\int_0^{\frac{\pi}{2}}\sin^9\theta\, d\theta = \frac{8\ \ 6\ \ 4\ \ 2}{9\ \ 7\ \ 5\ \ 3} = \frac{128}{315}$$

It is easily seen from the foregoing that

$$\int_0^{\frac{\pi}{2}}\cos^n x\,dx = \int_0^{\frac{\pi}{2}}\sin^n x\,dx$$

Definite integrals.—The following definite integrals are important in later work, particularly in dealing with vibrations and periodic movements

$$\int_{-\pi}^{\pi} a_0 \cos nx\,dx = a_0\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi}\{\cos(m+n)x + \cos(m-n)x\}dx$$

$$= \tfrac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi},$$

when m and n are unequal, $\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = 0$,

when $m = n$,

$$\int_{-\pi}^{\pi} \cos^2 nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi}(\cos 2nx + 1)dx = \tfrac{1}{2}\left[\frac{\sin 2nx}{2n} + x\right]_{-\pi}^{\pi}$$

$$= \tfrac{1}{2}\{0 + \pi - (-\pi)\} = \pi$$

Similarly, $$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0,$$

and $$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \pi$$

$$\int_{-\pi}^{\pi} \sin nx \cos nx\,dx$$

$$= \tfrac{1}{2}\int_{-\pi}^{\pi} \sin 2nx\,dx = \frac{1}{4n}\left[-\cos 2nx\right]_{-\pi}^{\pi} = 0$$

Fourier's series.—Assuming that between the limits π and $-\pi$,

$$f(x) = a_0 + a_1\cos x + a_2\cos 2x + \dots + a_n\cos nx + \dots$$
$$+ b_1\sin x + b_2\sin 2x + \dots + b_n\sin nx + \dots,$$

multiply through by $\cos nx$ and integrate Then,

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi} a_0\cos nx\,dx + \int_{-\pi}^{\pi} a_1\cos x \cos nx\,dx + \dots$$

$$+\int_{-\pi}^{\pi} a_n \cos^2 nx\,dx + \ldots \int_{-\pi}^{\pi} b_1 \sin x \cos nx\,dx$$

$$+\int_{-\pi}^{\pi} b_n \cos nx \sin nx\,dx + \ldots.$$

At the limits, the only term on the right which does not vanish is

$$\int_{-\pi}^{\pi} a_n \cos^2 nx\,dx = \pi a_n\,,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx \quad \ldots \quad \ldots \quad \ldots \text{(i)}$$

Taking as an example,

$$y = f(x) = \Big[\pi + x\Big]_{-\pi}^{-\frac{\pi}{2}}$$

$$+\Big[-x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \Big[x - \pi\Big]_{\frac{\pi}{2}}^{\pi} + \text{etc}$$

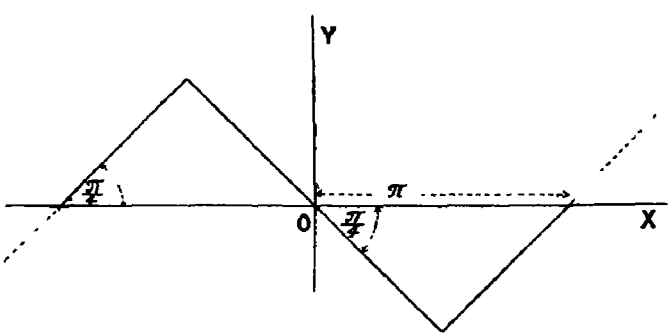

FIG 167

A function of x which has

the value $(\pi + x)$ from $x = -\pi$ to $-\frac{\pi}{2}$,

the value $\quad -x$ from $x = -\frac{\pi}{2}$ to $+\frac{\pi}{2}$,

the value $(x - \pi)$ from $x = \frac{\pi}{2}$ to π, and so on.

In this case $\qquad a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{-\frac{\pi}{2}}(\pi+x)\cos nx\,dx-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\cos nx\,dx+\int_{\frac{\pi}{2}}^{\pi}(x-\pi)\cos nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\frac{n(x+\pi)\sin nx+\cos nx}{n^2}\right]_{-\pi}^{-\frac{\pi}{2}}-\left[\frac{nx\sin nx+\cos nx}{n^2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right.$$
$$\left.+\left[\frac{n(x-\pi)\sin nx+\cos nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}\right\}$$

$$=\frac{1}{\pi n^2}\left\{-\frac{n\pi}{2}\sin\frac{\pi n}{2}+\cos\frac{n\pi}{2}-\cos n\pi-\left(\frac{n\pi}{2}\sin\frac{n\pi}{2}-\frac{n\pi}{2}\sin\frac{n\pi}{2}\right)\right.$$
$$\left.+\frac{n\pi}{2}\sin\frac{n\pi}{2}+\cos n\pi-\cos\frac{n\pi}{2}\right\}$$

$$=0.$$

There are therefore no cosine terms in the expansion,

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{-\frac{\pi}{2}}(\pi+x)\sin nx\,dx-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\sin nx\,dx+\int_{\frac{\pi}{2}}^{\pi}(x-\pi)\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\frac{\sin nx-n(x+\pi)\cos nx}{n^2}\right]_{-\pi}^{-\frac{\pi}{2}}-\left[\frac{\sin nx-nx\cos nx}{n^2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right.$$
$$\left.+\left[\frac{\sin nx-n(x-\pi)\cos nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}\right\}$$

$$=\frac{1}{\pi n^2}\left\{-\sin\frac{n\pi}{2}-2\sin\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right\}$$

$$=-\frac{4\sin\frac{n\pi}{2}}{\pi n^2}.$$

This is equal to zero for all even values of n, and to $-\dfrac{4}{\pi n^2}$ for all terms of the type $(4r+1)$, and to $\dfrac{4}{\pi n^2}$ for all terms where n is of the type $(4r-1)$; finally we have

$$f(x)=-\frac{4}{\pi}\left\{\sin x-\frac{1}{9}\sin 3x+\frac{1}{25}\sin 5x-\frac{1}{49}\sin 7x+\text{etc}\ ..\right\}.$$

If we put $x = -\frac{\pi}{2}$, $f(x)$ when calculated from the first 16 terms is equal to 1 55, or very approximately $\frac{\pi}{2}$

Putting $x = -\frac{\pi}{4}$ and using 16 terms, we obtain

$$f(x) = 0\,78 \text{ or } \frac{\pi}{4}.$$

Fourier's theorem —This important theorem states that any periodic function $f(x)$ may be fully represented by the sum of a constant term and a series of sines and cosines of multiples of that variable, and may be expressed in the form

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots \text{ etc}$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \quad \text{etc},$$

in which the second S H M would have one-half the period of the preceding one, the next one-third, and so on

The theorem may be written in the form

$$f(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \text{etc}$$
$$= a_0 + a_{11_1} \sin(x + \alpha_1) + a_{11_2} \sin(2x + \alpha_2) + a_{11_3} \sin(3x + \alpha_3) + \ldots,$$

where $a_{11_1} = \sqrt{a_1^2 + b_1^2}$ and $\tan \alpha_1 = \frac{a_1}{b_1}$, etc

The series now becomes

$$f(x) = c_0 + a \sin(x + \alpha) + b \sin(2x + \beta) + c \sin(3x + \gamma) +$$

If we divide the period into n equal parts and superpose the parts, we get the constant c_0 increased to nc_0

Taking the case $y = a \sin(x + \alpha)$ and omitting the case $n = 1$, we obtain on superposing

$$\Sigma y = a\left\{\sin \alpha_1 + \sin\left(\alpha_1 + \frac{2\pi}{n}\right) + \sin\left(\alpha_1 + \frac{4\pi}{n}\right) + \text{etc}, \text{ to } n \text{ terms}\right\}$$

(where $\alpha_1 = \alpha + x$ is the x of the first point taken)

$$= a \frac{\sin\left(\alpha_1 + \frac{n-1}{n} \cdot \pi\right) \sin \pi}{\sin \frac{\pi}{n}},$$

which is zero for all values of n greater than 1

Therefore, we may split up the given curve into n equal parts, and on superposing we shall find the fundamental vibration to be eliminated.

Now take $y=b\sin(2x+\beta)$.

$$\Sigma y=b\left\{\sin\gamma_1+\sin\left(\gamma_1+\frac{4\pi}{n}\right)+\sin\left(\gamma_1+\frac{8\pi}{n}\right)+\text{etc}, \text{ to } n \text{ terms}\right\}$$

$$=b\frac{\sin\left\{\gamma_1+\left(\frac{n-1}{n}\right)2\pi\right\}\sin 2\pi}{\sin\frac{2\pi}{n}},$$

which is zero for all values of n, greater than 1, except $n=2$ when it becomes

$$\frac{-b\sin\gamma_1\sin 2\pi}{\sin\pi}=b\frac{-\sin\gamma_1 2\cos\pi\sin\pi}{\sin\pi}$$
$$=2b\sin\gamma_1$$

Therefore, when $n=2$ the fundamental is eliminated, whilst the octave, *i.e* the vibration with double frequency, remains, but becomes doubled in amplitude

In the same way, if $y=a\sin(mx+\beta)$, where m is a **prime number**, the superposition will cause the term to vanish for any other value of n than m, and for that particular value we get an expression of the same frequency, but of m times the amplitude.

Now, taking the case $m=12$,

$$y=a\sin(12x+a),$$

$$\Sigma y=a\left\{\sin a_1+\sin\left(a_1+\frac{24\pi}{n}\right)+\sin\left(a_1+\frac{48\pi}{n}\right)+\ \ .\text{ to } n \text{ terms}\right\}$$

$$=a\frac{\sin\left(a_1+\frac{n-1}{n}12\pi\right)\sin 12\pi}{\sin\left(\frac{12\pi}{n}\right)},$$

where, as before, a_1 is the first value of $12x+a$

This is equal to zero, except for the values $n=1$, $n=2$, $n=3$, $n=4$, $n=6$, and $n=12$

Taking the case $n=6$, we obtain

$$\Sigma y=a\frac{\sin(a_1)\sin 12\pi}{\sin 2\pi}=a\frac{\sin a_1 12\cos 12\pi}{2\cos 2\pi}=6a\sin a_1,$$

[The second step being obtained by the differential calculus rule for undetermined forms]

i.e. the amplitude is increased six times, the period remaining the same

Therefore the effect of dividing into n parts and superposing is to eliminate all terms excepting those of the form $a\sin(rnx+\alpha)$, where $rn=m$, which terms remain of the same periods, but are increased in amplitude n times.

From these results the following method of analysing a curve which represents some periodic motion, such as the movement of a piston or slide valve, is deduced.

Let the relation between x and θ be supposed to be

$$f(x)=a_0+a\sin(\theta+\alpha)+b\sin(2\theta+\beta)+c\sin(3\theta+\gamma)+\text{ etc}$$

In order to simplify the expressions write $s\theta$ instead of $a\sin(\theta+\alpha)$,

$$f(x)=a_0+s\theta+s2\theta+s3\theta+s4\theta+s5\theta+s6\theta+\text{ etc.}$$

Dividing into two and superposing, calling the result $f(x_2)$,

$$f(x_2)=2(a_0+s2\theta+s4\theta+s6\theta+s8\theta+\text{ etc}),$$

$$2f(x)-f(x_2)=2(s\theta+s3\theta+s5\theta+s7\theta+\text{ etc.})$$

Dividing into three parts and superposing,

$$f(x_3)=3(a_0+s3\theta+s6\theta+s9\theta+\text{ etc})$$

Similarly,

$$f(x_4)=4(a_0+s4\theta+s8\theta+s12\theta+\text{ etc}),$$

$$4f(x_3)-3f(x_4)=12(s3\theta-s4\theta+s6\theta-s8\theta+\text{ etc});$$

$$6\{2f(x)-f(x_2)\}-\{4f(x_3)-3f(x_4)\}$$
$$=12(s\theta+s4\theta+s5\theta-s6\theta+\text{ etc})$$

In this way we may eliminate the s-functions on the right, until the uneliminated terms after the first are so small that they may be neglected. We can now calculate the value of the fundamental, then, using this result, proceed to find, in a similar manner, the values of the other s-functions, one by one, so far as may be necessary; when this has been done the curve assumes the form of the sine-curve The distance between any two points of its intersection with the x-axis is a multiple of the period, π Measurement of the curve will give very approximately the constants in $Y=12a\sin(x+a_1)$. Thus, $12a$ is equal

to the average amplitude; if $x_0, x_1, x_2, \ldots x_n$ are the abscissae of the points of intersection of the curve and the x-axis, then

$$(n+1)\alpha_1+\frac{n(n+1)}{1 \cdot 2}\pi=x_0+x_1+\ldots+x_n,$$

from which α_1 may be determined

The student has now a choice of three methods of proceeding to determine each of the remaining terms in the Fourier's series:

(1) Determine $s2\theta$ by a process exactly similar to that adopted for $s\theta$.

(2) Subtract from the curve $f(x)$ the part $a\sin(x+\alpha_1)$ as previously found

(3) Subtract from the curve $f(x)$ a *new calculated* curve $a\sin(x+\alpha_1)$.

When the terms $s\theta$, $s2\theta$, $s3\theta$, etc., have been determined so far as found necessary, it is advisable to re-draw these curves and by adding the ordinates, in the usual manner, to determine the curve $s\theta+s2\theta+s3\theta+$etc Comparison of this (calculated) curve with the problem will give some idea as to the accuracy of the calculation and of the hypothesis of the *relative smallness of rejected terms*

If the form of the calculated curve is sufficiently near that of the problem curve, then there only remains to find k which is the vertical distance between the horizontal axes of the two curves.

If the two curves should be too much unlike, take the difference of the problem curve above the calculated one and proceed to a fresh calculation

The application of the theorem to a given curve (Fig 168) may be seen from the following example

Taking equal intervals $\frac{\pi}{6}$ for θ along the base OX and setting up the twelve ordinates,

$$f(\theta)=k+a\sin(\theta+\alpha)+b\sin(2\theta+\beta)+c\sin(3\theta+\gamma)+\text{ etc}$$
$$=k+s\theta+s2\theta+s3\theta+\text{ etc. (with the previous notation).}$$

Dividing into two and superposing,

$$f(\theta_2)=2(k+s2\theta+s4\theta+\text{ etc.}),$$
$$2f(\theta)-f(\theta_2)=2(s\theta+s3\theta+s5\theta+\text{ etc.});$$

when $\theta=0$, $\qquad f(\theta)=0,$

$$f(\theta_2)=0 \text{ to } 6;$$

which result simply means that to the ordinate passing through 0 must be added the ordinate passing through 6 Similarly, to the 1st ordinate add the 7th and so on, draw a fair curve through the points and obtain $f(\theta_2)$ The sum of the 0 and 6th ordinates is the particular ordinate here mentioned;

$\therefore$ ordinate of $(\theta, 3\theta, 5\theta-)=\frac{1}{2}(6 \text{ to } \theta)$;

when $\theta=\frac{\pi}{6}$, $\qquad f(\theta)=0 \text{ to } 1,$

$$f(\theta_2)=(0 \text{ to } 7)+(0 \text{ to } 1);$$

$$\text{ordinate of } (\theta, 3\theta, 5\theta)=\tfrac{1}{2}\{2(0 \text{ to } 1)-(0 \text{ to } 1)-(0 \text{ to } 7)\}$$
$$=\tfrac{1}{2}(7 \text{ to } 1)$$

In this way we get all the ordinates of $(\theta, 3\theta, 5\theta)$ as

$\frac{1}{2}$	6 to 0	0 to 6
	7 to 1	1 to 7
	8 to 2	2 to 8
	9 to 3	3 to 9
	10 to 4	4 to 10
	11 to 5	5 to 11

$$f(\theta_2)=2(k+2\theta+4\theta+6\theta+\text{etc.}).$$

Dividing into four parts and superposing,

$$f(\theta_4)=4(k+4\theta+8\theta+\text{ etc }),$$

$$\therefore 2f(\theta_2)-f(\theta_4)=4(2\theta+6\theta+10\theta+\text{ etc }),$$

when $\theta=0$, $\qquad f(\theta_2)=0 \text{ to } 6,$

$$f(\theta_4)=(0 \text{ to } 6)+(0 \text{ to } 3)+(0 \text{ to } 9),$$

$$\therefore \text{ ordinate of } (2\theta, 6\theta) \text{ is } \tfrac{1}{4}\{2(0 \text{ to } 6)-(0 \text{ to } 6)+(3 \text{ to } 0)+(9 \text{ to } 0)\}$$
$$=\tfrac{1}{4}(0 \text{ to } 6+3 \text{ to } 0+9 \text{ to } 0)$$
$$=\tfrac{1}{4}(3 \text{ to } 0,\ 9 \text{ to } 6)$$

The ordinates of the $(2\theta, 6\theta)$ are

$\frac{1}{4}$	3 to 0,	9 to 6	
	4 to 1,	10 to 7	
	5 to 2,	11 to 8	+ − these
	6 to 3,	0 to 9	
	7 to 4,	1 to 10	
	8 to 5,	2 to 11	

Proceeding in the same manner for $(3\theta, 9\theta)$, we obtain as ordinates

$\frac{1}{6}$	2 to 0, 6 to 4, 10 to 8
	3 to 1, 7 to 5, 11 to 9
	$-$
	$+$
	$-$
	$\pm$

Graphical method of harmonic analysis.—It has already been seen (p. 138) that motion in a straight line, which is compounded of two simple harmonic motions of the same period, is itself a simple harmonic motion of that period. The theorem may be represented by the equation

$$y = a \sin(qt + \alpha) + b \sin(qt + \beta) = A \sin(qt + E), \quad \ldots\ldots (\text{i})$$

where y is the displacement from mid-position at a time t

When the component motions a, b, α, β, are given, then for any given value of t, a parallelogram having a and b for its sides can be drawn, and the diagonal will give the amplitude, or radius, A, of the resultant motion As qt denotes the amount of turning, or angle, in radians it is convenient to write Eq (i) in the form

$$y = a \sin(\theta + \alpha) + b \sin(\theta + \beta) \quad \ldots\ldots\ \ldots\ldots \quad (\text{ii})$$

The parallelogram is inapplicable when the periods are different, but in such a case two sinuous curves may be separately drawn, and their ordinates added together will give the resultant curve

Thus, for example, the motion of the slide valve of a steam engine generally proves to be a close approximation to a simple harmonic motion The deviation from this fundamental motion usually consists of a small superposed octave, or a simple harmonic motion of comparatively small amplitude and of twice the frequency If y denotes the displacement of the valve from its mean position, the above Eq (ii) may be written

$$y = a \sin(\theta + \alpha) + b \sin(2\theta + \beta). \quad \ldots\ldots\ldots\ \ldots\ldots (\text{iii})$$

The diagrams of displacement consist of two sinuous curves, the first having an amplitude a and angular advance α, the amplitude and angular advance of the second being b and β respectively; the period of the second is one-half that of the first

Ex. 1. $y = 2\sin(\theta + 30°) + 0\ 5\sin(2\theta + 45°)$.

Let $y_1 = 2\sin(\theta + 30°)$ and $y_2 = 0{\cdot}5\sin(2\theta + 45°)$,

when $\theta = 0°$, $y_1 = 2\sin 30° = 1$;

and when $\theta = 30°$, $y_1 = 2\sin 60° = 1\ 73$.

In a similar manner from $y_2 = 0{\cdot}5\sin(2\theta + 45°)$,

when $\theta = 0°$, $y_2 = 0\ 5\sin 45° = \frac{\sqrt{2}}{4} = 0\ 35$;

and when $\theta = 30°$, $y_2 = 0\ 5\sin 105° = 0\ 5\sin 75° = 0\ 48$

Other values of θ may be assumed and the values of y_1 and y_2 calculated and tabulated as follows

Values of θ	0°	30°	60°	90°	120°
y_1	1	1 73	2	1 73	1
y_2	0 35	0 48	0 13	-0 35	-0 48
$y = y_1 + y_2$	1 35	2 21	2 13	1 38	0 52

Plot the values of y from the last row, and the curve passing through the plotted points will show the value of y for any value of θ.

Graphical method of composition.—The process may be easily carried out graphically as follows.

Draw a circle with centre C and radius 2″ (Fig 168); through C draw a horizontal line CL Make the angle MCA equal to 30° Divide the circle into 12 equal parts, and from any convenient point N on the line CL measure off 12 equal divisions from N to L, at each point draw the ordinates qp, fh, perpendiculars to CL Each of the equal divisions on the circle and on NL will denote 30°; number the points on the circle and on NL, 0, 1, 2, 11 as shown, then, points on the required curve can be found by projection, the projection through 0 on the circle cutting the ordinate through N at E etc In this manner the dotted curve (i) can be obtained.

Draw another circle with centre C and radius 0·5″, and make the angle $MCB = 45°$. As the point B rotates at twice

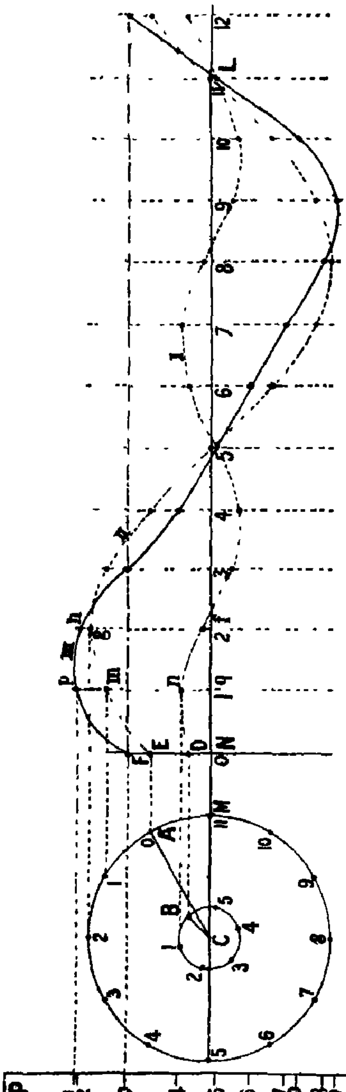

FIG. 168.—Graph of $y=2\sin(\theta+30^\circ)+0.5\sin(2\theta+45^\circ)$

the rate of A, it is only necessary to divide the circle into six equal parts, as shown in Fig 168.

By projecting as before the curve (ii) may be obtained. The final curve (iii) is obtained by adding the ordinates of the two curves at each point, thus,

$$qp = qm + qn,$$

i.e. by means of a pair of dividers, or the edge of a strip of paper, add qn to qm, and in this manner a series of points is determined; joining these by a fair curve the resultant curve (iii) is obtained.

The converse problem. Resolution.—The converse problem to obtain the elements of the component motions of a curve such as (iii) (Fig. 168) is of great importance. Such a curve is easily set out if the displacements, or ordinates, corresponding to given angular intervals are known. These may be marked on the edge of a strip of paper, or thin cardboard, as indicated at P (Fig. 168). For this purpose a line is drawn through the initial point

F, parallel to the base line NL. If y denotes the displacement, then, supposing the equation of the curve may be expressed by three terms of a Fourier's series, *i.e*

$$y=k+a\sin(\theta+\alpha)+b\sin(2\theta+\beta)$$

where k is the distance NF.

The analytical process by which the various constants in a Fourier's series are obtained is laborious and to some extent complicated By a simple graphical method, devised by Mr. J. Harrison, it will be found that any given curve can be readily analysed by merely using a strip of paper as follows:

Let $y=k+a\sin(\theta+\alpha)+b\sin(2\theta+\beta)+c\sin(3\theta+\gamma)+\ldots$

be a complete Fourier's series, which for shortness write

$$y=k+\theta+s2\theta+s3\theta+s4\theta, \quad .$$

Let the values of twelve equidistant ordinates, spread over the cycle, be denoted by $y_0, y_1, y_2, y_3 \ldots y_{11}$ From a fixed point on a strip of paper, set off these values along the edge, numbering the points 0, 1, 2, 3, 4, 5 ... 11. These points would represent twelve successive positions of a particle vibrating according to the above law. By employing the principle of superposition we arrive at the results given on p. 460.

The analysis of such a curve as that in Ex. 1 by using a paper strip may be seen from the following example.

Ex. 2. Twelve positions of a slide valve numbered 0, 1, 2, . 11, corresponding to intervals of 30° of the crank beginning at the inner dead point, are given in Fig 169 Analyse the motion so as to express the displacement of the valve from its mean position in the form

$$y=a\sin(\theta+a)+b\sin(2\theta+b),$$

θ being any crank position measured from the inner dead point. State the actual numerical values of a, b, a, and β in this case.

Mark off the given displacements along the edge of a strip of paper. On a sheet of squared paper mark off twelve equal horizontal distances and number these 0, 1, 2, 11, as in Fig. 169. Each of these equal divisions will denote 30°.

On the ordinate through 1, mark off from the paper strip the distance 01; similarly on the ordinate through 2 the distance 02, etc. Proceeding in this manner a series of points on

Table of Analysis.

The complete curve made up of k, θ, 2θ, 3θ, Call this series of ordinates *A*	Divide *A* into two equal parts, superpose and add. Some of the terms cancel, and there remain $2(k, 2\theta, 4\theta, 6\theta, \quad)$ *B*.	Divide *A* into three equal parts, superpose and add. The components remaining are $3(k, 3\theta, 6\theta, 9\theta, \quad)$ *C*.
y_0 y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 y_{10} y_{11}	y_0+y_6 y_1+y_7 y_2+y_8 y_3+y_9 y_4+y_{10} y_5+y_{11}	$y_0+y_4+y_8$ $y_1+y_5+y_9$ $y_2+y_6+y_{10}$ $y_3+y_7+y_{11}$
	Divide *B* into two equal parts, superpose and subtract, obtaining $4(2\theta, 6\theta, 10\theta, \quad)$ *E*	Divide *C* into two equal parts, superpose and subtract $6(3\theta, 9\theta, 15\theta, \quad)$ *F*
	$y_0+y_6-(y_3+y_9)$ $y_1+y_7-(y_4+y_{10})$ $y_2+y_8-(y_5+y_{11})$ that is $(y_0-y_3)+(y_6-y_9)$ $(y_1-y_4)+(y_7-y_{10})$ $(y_2-y_5)+(y_8-y_{11})$ or on the strip 0 to 3+6 to 9 1 to 4+7 to 10 2 to 5+8 to 11	$(y_0+y_4+y_8)-(y_2+y_6+y_{10})$ $(y_1+y_5+y_9)-(y_3+y_7+y_{11})$ that is $y_0-y_2+y_4-y_6+y_8-y_{10}$ $y_1-y_3+y_5-y_7+y_9-y_{11}$ or using the strip 0 to 2+4 to 6+8 to 10 1 to 3+5 to 7+9 to 11
Divide *A* into two equal parts, superpose and subtract. There result $2(\theta, 3\theta, 5\theta, \quad)$ *D* y_0-y_6 — 0 to 6 y_1-y_7 — 1 to 7 y_2-y_8 — 2 to 8 y_3-y_9 — 3 to 9 y_4-y_{10} — 4 to 10 y_5-y_{11} — 5 to 11 (or applying strip of paper inverted)		Deduct $\frac{1}{3}$ of ordinates of curve *F* from those of curve *D*. Then we obtain $D-\frac{F}{3}=2(\theta, 5\theta, 7\theta, 11\theta, \quad)$.

the curve of displacement is obtained and through these points a fair curve may be drawn.

To obtain the elements of the component motions *the strip is inverted.* Putting 0 on the strip to coincide with 0 on *BN*, mark off on the ordinate through 0, the distance 0 to 6. Similarly, putting 1 on the strip coincident with 1 on *BN*, set

off on the ordinate the distance 1 to 7. These processes may be written as 0 to 6, 1 to 7, 2 to 8, etc., as on p. 455. Draw a curve through the points.

Using the contracted notation the equation of the new curve may be written in the form $2(\theta, 3\theta, 5\theta\ .\ )$.

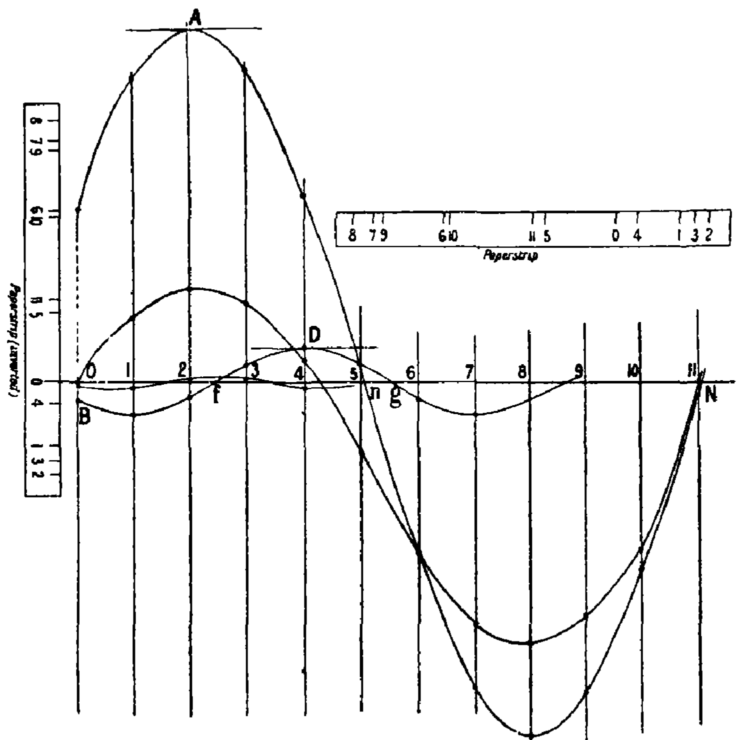

FIG 169 —Analysis of a displacement curve.

Draw a tangent to this curve at a maximum or minimum point; then the amplitude a is one-half the distance from A to the base line BN, or $6{\cdot}28-2=3{\cdot}14$

The magnitude of the angle α can be obtained by producing the curve to cut the line BN, then, as the distance nN denotes 180°, the distance $0n$ is proportionately $=151^\circ{\cdot}8$.

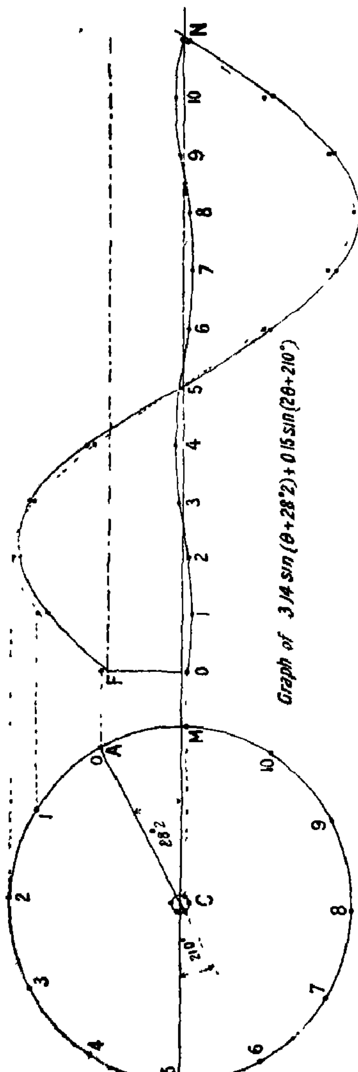

Fig 170

To obtain the angle a it is only necessary to subtract 151°·18 from 180°;

$\therefore\ a = 28°{\cdot}2.$

To obtain the elements of the second term with the strip inverted (*i.e* in the same position as before) make 0 on the strip to coincide with 0 on $0N$ Along the ordinate through 0, mark off a distance 0 to 3. Make the point 6 on the strip to coincide with this point and measure the distance 6 to 9. Then, this point gives a point on the required curve The process just described may be expressed as (0 to 3) + (6 to 9) A second point is determined by using the strip on point 1, *i e* (1 to 4) + (7 to 10), etc In a similar manner other points may be determined as indicated on p 455 Finally, draw the curve through the points so obtained The value of b is obtained by drawing the tangent at a maximum or minimum point as at D (Fig 169), and dividing the distance between the tangent and the line $0N$ by 4, *i e*.

$$b = \frac{0\ 6}{4} = 0{\cdot}15''$$

It will be noticed that the distance fg between the two points where the curve

intersects the line $0N$, corresponds to three divisions, hence, each division is 60°. The value of β could be found by producing the curve until the position in the positive direction was obtained, but the value may also be found by noting that the distance between the line passing through 0 and the point f, where the curve intersects the axis, is 2 5 divisions or 150°.

$$\beta = 360° - 150° = 210°.$$

To find the elements of the third term it would be necessary to proceed in a similar manner, viz to use the inverted strip and mark off from 0 a point corresponding to the distance 0 to 2; then to shift the strip so that 4 on the strip coincides with the point, and to mark off the distance 4 to 6. Finally, to put 8 at the last point and mark off the distance 8 to 10. The process just described is conveniently written in the form (0 to 2) + (4 to 6) + (8 to 10). Similarly, for the next point we should have (1 to 3) + (5 to 7) + (9 to 11) Proceeding in this manner a series of points is obtained The curve passing through the plotted points will be expressed, with the present notation, by 6 (3θ, 9θ, 15θ) In the present case this practically coincides with the line $0N$, merely showing a slight ripple; also, as the distance from the crest of the curve to the line $0N$ must be divided by 6 to give the magnitude of the amplitude a, it is obvious that the third term in the series is negligible Hence the equation may be written

$$y = 3{\cdot}14 \sin(\theta + 28° 2) + 0{\cdot}15 \sin(2\theta + 210°). \quad \text{.. (iii)}$$

It is instructive to reverse the process and obtain, as in Ex. 1, the curve represented by Eq (iii). Thus, draw a circle radius 3·14 (Fig 170) and set off an angular advance MCA of 28° 2; also draw a circle concentric with the former, radius 0·15, and set off an angular advance of 210°; project as already described in Ex. 1 Finally, add the ordinates of the two curves. It will be found that the resulting curve will be the same as the given one This result may be tested by using a piece of tracing paper, or by the paper strip. In the latter case, it is necessary to draw a line through the initial point F parallel to CN. The distance $0F$ is the value of the constant $k = 1$ 4. Hence, referred to axes of co-ordinates passing through F, the required equation is

$$y = -1{\cdot}4 + 3\ 14 \sin(\theta + 28° 2) + 0{\cdot}15 \sin(2\theta + 210°)$$

EXERCISES XLIV.

Integrate the following

1. $\sin ax \sin bx\,dx$

2. $x^2 \cos x\,dx$

3. $\dfrac{x\,dx}{(x-a)(x-b)}$

4. $\dfrac{(x^2-5x+7)\,dx}{x^2-5x+6}$.

5. $\dfrac{(x^2+7)\,dx}{x^4+5x^2+4}$

6. $\dfrac{(x^3+2x+4)\,dx}{x^3+2x^2+4x+8}$

7. $x^3(\log x)^2 dx$

8. $\dfrac{x^3 dx}{(x-a)(x-b)(x-c)}$

9. $\dfrac{x\,dx}{(x-3)^2(x+2)}$

10. $\theta \sin\theta\,d\theta$

11. $x^3 \cos x\,dx$.

12. $\dfrac{(2x-5)\,dx}{(x+3)(x+1)^2}$

13. $\dfrac{(6x^2+13x-43)\,dx}{x^3-13x-12}$

14. The motion of a point in a straight line is compounded of two simple harmonic motions of nearly equal periods, represented by the equation

$$x = 2\cdot 1 \sin\left(9t + \frac{\pi}{4}\right) + \sin 8t,$$

where x is the displacement in inches from the mean position, and t is time

Let the complete period of the vibration be divided into nine equal intervals Taking only the first, fourth, and seventh of these intervals, in each case draw a curve in which abscissae shall represent times, and ordinates the corresponding displacements of the point

Let the time of one of the intervals be represented on the paper by a length of 8″. In determining successive ordinates, the method of projection from the resultant crank may be used with advantage.

15. The displacements of a slide valve actuated by a Gooch link were measured at eight intervals each of 45°, and found to be as follows, beginning with the crank on the inner dead centre

2·44″, 1·65″, 0, −1·37″, −1·87″, −1·37″, 0, 1·65″

Assuming that the motion of the valve is compounded of two simple harmonic motions, one of double the frequency of the other, as represented by the equation

$$y = k + a\sin(\theta + \alpha) + b\sin(2\theta + \beta),$$

where θ is the crank angle. Find the values of k, a, α, b, β

CHAPTER XXII.

DIFFERENTIAL EQUATIONS.

Differential equations.—Any equation which connects the variables x and y, and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ etc., is called a **differential equation.** Such equations are of great importance. It will be found, for instance, that the majority of the so-called "laws" in dynamics, etc., can be expressed in their most general form by means of such equations.

It is only possible to give a few of the simpler cases; for further information the student is referred to larger books, such as that of Dr. Forsyth.

A simple form is furnished by the equation

$$y = a + \frac{dy}{dx}x \quad \ldots \ldots \ldots \quad \text{(i)}$$

The relation expressed by (i) represents a series of straight lines making an intercept a on the axis of y and having slopes $\frac{dy}{dx} = \tan\theta$ (Fig. 171).

From (i) we obtain

$$y - a = x\frac{dy}{dx},$$

or,

$$\frac{dy}{y-a} = \frac{dx}{x}.$$

Integrate each side;

$$\log(y-a) = \log bx,$$

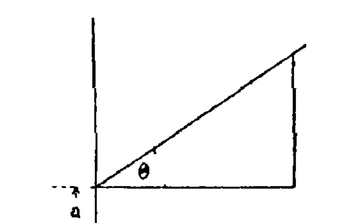

FIG. 171

or

$$y = a + bx. \ldots \ldots \ldots \quad \text{(ii)}$$

From (ii) $y=a+bx$, the equation to a straight line, we obtain by differentiation

$$\frac{dy}{dx}=b$$

Hence, we see that b simply denotes the inclination of the line to the axis of x, or, shortly, the **slope** of the line

Again, from (ii), $\frac{d^2y}{dx^2}=0$. As both the constants have been eliminated, this is the most general equation of a straight line.

Ex 1 Given $$\frac{dy}{dx}=b$$

This may be written $dy=bdx$

Integrating, $$\int dy=b\int dx\,;$$

$$y=bx+C. \qquad \text{(iii)}$$

This equation denotes a family of straight lines with constant slope. *As already indicated, any constant connected by the* signs $+$ and $-$ disappears during differentiation, and therefore a constant denoted by C is added to the indefinite integral to give the most general value to it It will be noticed that it is unnecessary to add a constant to both sides of the equation

Elimination of constants.—One, two, or more constants may be eliminated from a given equation by introducing $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, etc

Ex 2 Eliminate the constants a and b from the equation

$$y-ax^2+b=0 \qquad (1)$$

From (1) $$y=ax^2-b, \qquad \text{(i)}$$

$$\frac{dy}{dx}=2ax, \qquad \text{(ii)}$$

$$\frac{d^2y}{dx^2}=2a. \qquad \text{(iii)}$$

Divide (ii) by x and subtract from (iii):

$$\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}=0$$

The general method of eliminating two arbitrary constants may be carried out as follows ·

Let
$$y=f(a, b, x),$$
$$\frac{dy}{dx}=f'(a, b, x),$$
$$\frac{d^2y}{dx^2}=f''(a, b, x),$$
three equations from which to eliminate a and b.

Generally, to eliminate n constants it is necessary to use the first n differential coefficients, and, conversely, a differential equation of the n^{th} order requires for its solution n independent constants

Ex 3 Given
$$y^2=ax+bx^2, \qquad (i)$$
eliminate the constants a and b

Differentiating (i),
$$2y\frac{dy}{dx}=a+2bx \qquad (ii)$$

Again differentiating,
$$2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=2b. \qquad (iii)$$

Between (i), (ii), and (iii) eliminate a and b, therefore multiply (ii) by x and subtract from (i),
$$y^2-2xy\frac{dy}{dx}+bx^2=0$$

Substitute for b from (iii),
$$y^2-2xy\frac{dy}{dx}+x^2y\frac{d^2y}{dx^2}+x^2\left(\frac{dy}{dx}\right)^2=0$$

Ex. 4. If the letters s, v, and t denote space, velocity, and time respectively, then
$$v=\frac{ds}{dt} \text{ and acceleration}=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

If the relation between s and t is expressed by
$$s=at^2+bt+c, \qquad (i)$$
the average velocity is obtained from $\frac{\delta s}{\delta t}$ and the actual value is given by $\frac{ds}{dt}$ Thus, from (i)
$$\frac{ds}{dt}=2at+b \text{ and } \frac{d^2s}{dt^2}=2a$$

Hence, the acceleration is constant and equal to $2a$

Thus, let $a=\frac{1}{2}g$, $b=V$, and $c=0$, then Eq. (i) becomes by substitution the well-known formula

$$s=\tfrac{1}{2}gt^2+Vt;$$

$$\cdot\quad v=\frac{ds}{dt}=gt+V,$$

and $$\text{acceleration}=f=\frac{d^2s}{dt^2}=g;$$

therefore the acceleration is constant and equal to g.

As a simple example consider the differential equation

$$\frac{d^2s}{dt^2}=g$$

This denotes that the acceleration of a moving body is g

Integrating, $$\frac{ds}{dt}=v=gt+C$$

To determine the value of the constant C it is only necessary to know the value of v when $t=0$ Let this be V

Then, $$v=\frac{ds}{dt}=gt+V \quad \ldots \quad (\text{i})$$

Integrating again, $s=\frac{1}{2}gt^2+Vt+C_1$

If $s=0$ when $t=0$, then $C_1=0$;

$$s=\tfrac{1}{2}gt^2+Vt \quad \ldots \quad (\text{ii})$$

Obviously in (ii) the direction of the acceleration and the initial velocity are both vertically downwards, if V is upwards, then the space described in any time t is given by

$$s=Vt-\tfrac{1}{2}gt^2$$

From the relation Force = mass × acceleration,

$$F=m\frac{dv}{dt}=m\frac{d^2s}{dt^2} \quad \ldots \quad (\text{iii})$$

The work done by the force F through a distance ds is $F\ ds$;

$$\therefore \text{ from (iii) } F\ ds=m\frac{d^2s}{dt^2}ds=mv\,dv=m\frac{ds}{dt}\frac{d^2s}{dt^2}dt$$

Hence $$F\int ds=m\int v\,dv,$$

or $$Fs=\tfrac{1}{2}mv^2+C. \quad \ldots \quad (\text{i})$$

If when $s=0$, $v=0$, then (i) becomes $Fs=\frac{1}{2}mv^2$.

„ $s=0$, $v=u$, „ (i) becomes $Fs=\frac{1}{2}m(v^2-u^2)$

Ex. 5. Two unequal weights of 2 and 3 lbs. respectively are fastened to the ends of a string passing over a smooth pulley (Fig. 172). The equation of motion is

$$(M+m)\frac{d^2s}{dt^2}=(M-m)g$$

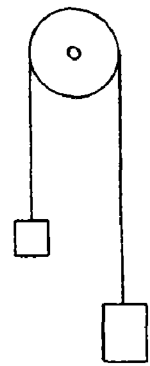

FIG 172.

Find the equation of motion if one weight is 3 ft from the ground and is moving with a velocity of 2 ft per sec. at the given instant Also find the position and velocity one second later, the time which has elapsed since starting from rest, and the position of the weight ($g=32{\cdot}2$).

From the relation

$$a=\text{acceleration}=\frac{\text{force causing motion}}{\text{mass moved}}\times g$$

we obtain $$\frac{d^2s}{dt^2}=\frac{1}{5}g,$$

$$v=\tfrac{1}{5}gt+C. \qquad \text{(i)}$$

Now $v=2$ when $t=0$; $\quad 2=C$,

or $$v=\tfrac{1}{5}gt+2=6{\cdot}44t+2. \qquad \text{(ii)}$$

Also $$s=\tfrac{1}{10}gt^2+2t+C_1$$

But $s=3$ when $t=0$,

$$s=\tfrac{1}{10}gt^2+2t+3=3{\cdot}22t^2+2t+3 \qquad \text{(iii)}$$

Put $t=1$, $\quad s=8{\cdot}22$ ft, from (iii),

and $\quad v=8{\cdot}44$ ft per sec., from (ii).

When $v=0$, $$t=-\frac{2}{6{\cdot}44}=-\frac{1}{3{\cdot}22}$$
$$=-0{\cdot}310,$$

position of the weight is then given by

$$s=3{\cdot}22\times0{\cdot}0961-0{\cdot}620+3$$
$$=2{\cdot}690.$$

Simple differential equations.—The following are a few of the more commonly recurring simple differential equations

Type 1. $$\frac{d^2y}{dx^2}=A+Bx+Cx^2+Dx^3+\ldots$$

The solution is

$$y=k+k_1x+\frac{Ax^2}{1.2}+\frac{Bx^3}{1.2.3}+\frac{Cx^4}{3.4}+\text{etc},$$

where k, k_1 are constants of integration.

As already indicated, p. 334, when a curve is very flat and parallel to the axis of x, we may use, instead of the more accurate expression for the curvature, the form $\frac{d^2y}{dx^2}$. Hence, the preceding result may be applied to problems dealing with the deflection of beams.

Cantilever with concentrated load at the free end.—Let l denote the length of the beam (Fig 173), and x the distance of a section from the fixed end, and y the deflection below the horizontal; then, the bending moment at such a section is $M=W(l-x)$,

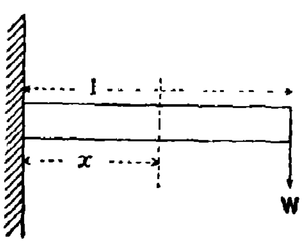

FIG 173 —Cantilever with concentrated load

$$\frac{d^2y}{dx^2}=\frac{W}{EI}(l-x)$$

Where

E=modulus of elasticity,

I=moment of inertia,

and y is measured downwards Integrating,

$$\frac{dy}{dx}=\frac{W}{EI}\int(l-x)$$

$$=\frac{W}{EI}\left(lx-\frac{x^2}{2}\right)+C$$

To find the value of the arbitrary constant C, we notice that, when $x=0$, $\frac{dy}{dx}=0$, $\quad C=0$

Again integrating,

$$y=\frac{W}{EI}\int\left(lx-\frac{x^2}{2}\right);$$

$$y=\frac{W}{EI}\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)+C_1.$$

Again, when $x=0$, $y=0$; $\quad C_1=0$

Hence,
$$y=\frac{W}{EI}\left(\frac{lx^2}{2}-\frac{x^3}{6}\right) \qquad (1)$$

In practical cases the maximum value of y is required, and this obviously occurs when $x=l$. Substitute this value in (1),

$$y=\frac{W}{EI}\left(\frac{l^3}{2}-\frac{l^3}{6}\right)=\frac{1}{3}\frac{Wl^3}{EI}.$$

Cantilever with uniform load.—If l denote the length of the beam (Fig 174), and w the load per unit length of the beam, the bending moment at a section distant x from the fixed end and y measured downwards

$$=w(l-x)\frac{(l-x)}{2}=\frac{w}{2}(l-x)^2;$$

$$\therefore \frac{d^2y}{dx^2}=\frac{w}{2EI}(l^2-2lx+x^2)$$

Integrating,

$$\frac{dy}{dx}=\frac{w}{2EI}\int(l^2-2lx+x^2)$$

$$=\frac{w}{2EI}\left(l^2x-\frac{2lx^2}{2}+\frac{x^3}{3}\right)+C$$

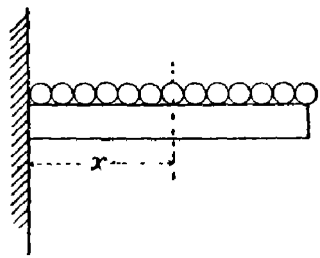

FIG 174 —Cantilever with uniform load

To obtain the numerical value of the arbitrary constant C we notice that $\frac{dy}{dx}=0$ when $x=0$, $\quad C=0$

Integrating again,

$$y=\frac{w}{2EI}\int\left(l^2x-lx^2+\frac{x^3}{3}\right)$$

$$=\frac{w}{2EI}\left(\frac{l^2x^2}{2}-\frac{lx^3}{3}+\frac{x^4}{12}\right)+C_1.$$

As in the preceding case, $y=0$ when $x=0$. $\quad C_1=0$

Hence,

$$y=\frac{w}{2EI}\left(\frac{l^2x^2}{2}-\frac{lx^3}{3}+\frac{x^4}{12}\right) \quad . \quad . \quad (1)$$

The maximum value of y obviously occurs when $x=l$.

Substituting this value in (1), we obtain

$$y=\frac{wl^4}{8EI},$$

or, if W denote the total load $=wl$,

then, $$y=\frac{1}{8}\frac{Wl^3}{EI}.$$

Beam supported at each end and loaded uniformly.—Let AB (Fig. 175) denote a beam carrying a uniform load of magnitude w per unit length; if l denotes the length of the beam, the total load will be wl

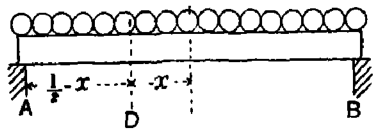

FIG 175—Beam supported at each end, uniform load

Take the origin at the middle of the beam Let D be a section at a distance x from the origin, y, measured downwards, then the bending moment at D is

$$-\frac{wl}{2}\left(\frac{l}{2}-x\right)+\frac{w}{2}\left(\frac{l}{2}-x\right)^2=-\frac{w}{2}\left(\frac{l^2}{4}-x^2\right);$$

$$\frac{d^2y}{dx^2}=-\frac{1}{EI}\times\frac{w}{2}\left(\frac{l^2}{4}-x^2\right),$$

or, integrating, $$\frac{dy}{dx}=-\frac{1}{EI}\frac{w}{2}\left(\frac{l^2x}{4}-\frac{x^3}{3}\right)+C,$$

when $x=0$, $\frac{dy}{dx}=0$, C is 0

Again, integrating,

$$y=-\frac{1}{EI}\frac{w}{2}\left(\frac{l^2x^2}{8}-\frac{x^4}{12}\right)+C_1.$$

Since, when $x=\frac{l}{2}$, $y=0$; $\therefore$ $C_1=\frac{5}{384}\frac{wl^4}{EI}$;

$$y=-\frac{w}{2EI}\left(\frac{l^2x^2}{8}-\frac{x^4}{12}\right)+\frac{5}{384}\frac{wl^4}{EI}$$

The maximum value of y occurs at the middle of the beam, i.e where $x=0$

Substituting this value for x, we obtain

$$y=\frac{5wl^4}{384EI}=\frac{5}{384}\frac{Wl^3}{EI},$$

where $W=wl$.

Beam fixed at both ends loaded with a uniform load.—Let w be the load per unit length, and l the length of the beam. The forces at one end, such as at A (Fig 176), consists of a shearing force $\frac{wl}{2}$, and a couple which may be denoted by C. Then, for a section at a distance x from A and y, measured downwards,

$$\frac{d^2y}{dx^2}=\frac{1}{EI}\left(C-\frac{wl}{2}x+\frac{wx^2}{2}\right),$$

$$\frac{dy}{dx}=\frac{1}{EI}\left(Cx-\frac{wlx^2}{4}+\frac{wx^3}{6}\right)+A_1$$

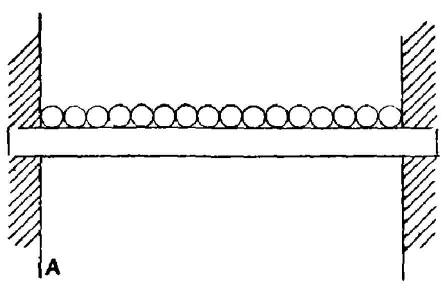

FIG 176 —Beam fixed at both ends, uniform load

To obtain the numerical values of the constants C and A_1, we notice that when $x=0$, $\frac{dy}{dx}=0$, $A_1=0$ When $x=l$, $\frac{dy}{dx}$ is again 0;

$$0=\frac{1}{EI}\left(Cl-\frac{wl^3}{4}+\frac{wl^3}{6}\right),$$

$$C=\frac{1}{12}wl^2$$

The equation becomes

$$\frac{dy}{dx}=\frac{1}{EI}\left(\frac{wl^2x}{12}-\frac{wlx^2}{4}+\frac{wx^3}{6}\right)$$

Again, by integration,

$$y=\frac{1}{EI}\left(\frac{wl^2x^2}{24}-\frac{wlx^3}{12}+\frac{wx^4}{24}\right)+A_1;$$

when x is 0, y is 0; $A_1=0$,

$$\therefore y=\frac{w}{24EI}(lx-x^2)^2.$$

Also y is maximum when $x=\frac{l}{2}$, substituting this value,

$$y=\frac{1}{384}\frac{wl^4}{EI}$$

That is to say, the deflection of a beam fixed at the ends is only $\frac{1}{5}$th of a similar beam the ends of which are merely supported.

Compound interest law.—A class of functions of great importance, such as e^x, e^{-x}, etc, is known as **exponential functions.** The base of such a function is, as indicated, usually taken to be e, the base of the Napierian logarithms When another base is used, such as in a^x, it may, if necessary, be expressed as e^{kx}, where k is a constant equal to $\log_e a$ In a general form the function may be written

$$y=Ae^{kx} \text{ or } y=Ae^{-kx}, \quad \text{(i)}$$

the former when the function is increasing, the latter when it is diminishing in magnitude

Many processes follow the laws given by Eq (i), and it has been very aptly styled by Lord Kelvin the **Compound Interest Law**

Money lent at compound interest increases in this way, and forms one of the simplest applications of this law Thus, if £100 is lent at 5 per cent per annum compound interest, then at the end of the first year the principal and interest amount to £105 This amount is the principal for the second year, and the interest will be charged on £105 instead of on £100, similarly, for the third year, etc. The preceding facts are better expressed symbolically as follows. Let P_0 denote the sum lent at r per cent per annum, then P_1, the principal for the second year, may be obtained from

$$P_1=P_0\left(1+\frac{r}{100}\right) \quad \text{(ii)}$$

The principal P_2 at the end of the second year is given by

$$P_2=P_1\left(1+\frac{r}{100}\right).$$

Substitute the value of P_1 from (ii) and this becomes

$$P_2 = P_0\left(1 + \frac{r}{100}\right)^2.$$

Similarly, at the end of the third year

$$P_3 = P_0\left(1 + \frac{r}{100}\right)^3$$

Hence, in t years $P_t = P_0\left(1 + \frac{r}{100}\right)^t$

If instead of adding the interest by annual increments the interest is added monthly, then at the end of t years the principal or amount A is given by

$$A = P_0\left(1 + \frac{r}{12 \times 100}\right)^{12t}.$$

Again, if instead of at monthly intervals, the interest is added at n equal intervals in each year, then in t years

$$A = P_0\left(1 + \frac{r}{n \times 100}\right)^{nt} \tag{i}$$

As the number n is increased, the interval of time t becomes shorter and shorter, and if n be indefinitely great the interest would be added continuously to the principal.

If $n = \frac{rm}{100}$ Eq (i) may be written

$$A = P_0\left\{\left(1 + \frac{1}{m}\right)^m\right\}^{\frac{rt}{100}}. \quad \ldots \quad \ldots \text{(ii)}$$

In the limit when n and therefore m become indefinitely great, Eq (ii) becomes

$$A = P_0 e^{\frac{rt}{100}}$$

[The value of $\left(1 + \frac{1}{m}\right)^m$ when m is indefinitely great is, on p 289, shown to be equal to e]

This result may be obtained in a more direct manner as follows

If P be the principal at the end of t years, then for a small increment of time, denoted by δt, the corresponding increment of P may be denoted by δP.

$$\delta P = \frac{r}{100} P\,\delta t, \quad \text{or,} \quad \frac{\delta P}{\delta t} = \frac{r}{100} P$$

Hence, when the interval of time is made indefinitely small,

$$\frac{dP}{dt} = \frac{r}{100} P;$$

$$\frac{dP}{P} = \frac{r}{100} dt$$

Integration gives $\log_e P = \frac{r}{100} t + C$

Now, since when $t=0$, $P=P_0$, where P_0 is the principal at the time 0, the constant is $\log_e P_0$,

$$\therefore P = P_0 e^{\frac{r}{100}t}$$

Write $e^{\frac{r}{100}} = e^k$, and the preceding result will become

$$P = P_0 e^{kt} \quad \ldots\ldots \quad (\text{ii})$$

Friction of a cord or belt on a pulley or cylinder.—Let $ANMB$ (Fig. 177) represent a belt or cord pressed tightly against a surface by forces at its free ends. Then, when the belt is just about to slip on the surface in the direction B to A, the tension at A is greater than at B. The angle AOB may be denoted by θ. Also MN may be taken to be a small portion of AB acted on by the tensions T at M, and $T+\delta T$ at N. Constructing the triangle of forces ABC (Fig. 177), it is readily seen that the radial force

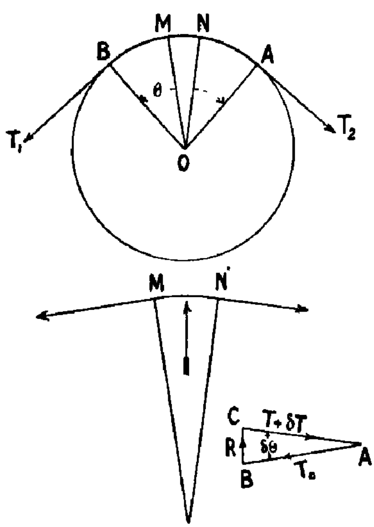

FIG 177

$$R = (T+\delta T)\delta\theta$$

Also friction $= \mu R$, where μ is constant.

Let R denote the reaction of the cylinder, then, resolving tangentially,

$$\frac{dT}{ds}+\mu R=0\,, \qquad \text{(i)}$$

resolving normally, $$T\frac{d\theta}{ds}-R=0 \qquad \text{(ii)}$$

Eliminating R we have $\dfrac{dT}{T}=\mu d\theta$

This is the compound interest law

Integration between the limits T_1 and T_2 of T and 0 and θ of θ, gives

$$\int_{T_1}^{T_2}\frac{dT}{T}=\mu\int_0^\theta d\theta,$$

or $$\log T_2-\log T_1=\mu\theta, \qquad \text{(iii)}$$

$$\frac{T_2}{T_1}=e^{\mu\theta} \qquad \text{(iv)}$$

If b denotes the width, and t the thickness of a belt, then the area of cross-section is calculated for the maximum tension T_2 with a margin for safety It will be noticed that when θ, the angle of contact of the belt with the cylinder, and the coefficient of friction μ are known, the ratio of T_2 to T_1 can be calculated from (iv) [The value of t for a single leather belt is usually about $\frac{3}{8}$ inch and the safe stress about 300 to 350 lbs per sq in]

Ex 6 A rope passes three times round a post and is held by a force of 10 lbs at one end What pull at the other end will be necessary to cause the rope to slip, assuming the coefficient of friction μ to be 0 3?

Here, if T_2 denote the force required,

$$\frac{T_2}{T_1}=e^{\mu\theta}=e^{0\,3\times 6\pi},$$

$$\log T_2=0\,3\times 6\pi\log 2\,718+\log T_1$$

$$=5\,656\times 0\,4343+1=3\,4564=\log 2860\,,$$

$$\therefore\ T_2=2860 \text{ lbs.}$$

An electrical example.—If V is the voltage, R the resistance of an electrical circuit in ohms, C the current in ampères,

then for a constant current Ohm's law, $V=RC$, applies, but when the current is not constant the law becomes

$$V=RC+L\frac{dC}{dt}, \qquad \ldots \ldots (1)$$

$\frac{dC}{dt}$ is the rate of increase of C, and L is called the self-induction of the circuit

If in (1) $V=0$, then

$$0=RC+L\frac{dC}{dt},$$

or

$$\frac{dC}{dt}=-\frac{R}{L}C,$$

$$\frac{dC}{C}=-\frac{R}{L}dt$$

Integrating,

$$\log C=-\frac{R}{L}t+K,$$

where K is a constant

To find the value of K, let C_0 be the value of C when $t=0$, then

$$\log C_0=0+K,$$

$$K=\log C_0$$

Hence, substituting,

$$\log\frac{C}{C_0}=-\frac{R}{L}t,$$

$$C=C_0e^{-\frac{R}{L}t},$$

again the compound interest law

Whence $$V=RC-RC_0e^{-\frac{R}{L}t}.$$

Hence, as t increases, the effect of a constant self-induction decreases

Ex. 7. The current C ampères in a circuit follows the law, $C=10\sin 600t$; if t is in seconds, and if

$$V=RC+L\frac{dC}{dt}, \qquad (1)$$

where R is 0 3, and L is 4×10^{-4}, what is V?

From the relation $C=10\sin 600t$ we find

$$\frac{dC}{dt}=6000\cos 600t$$

Hence, substituting in (i),

$$V=0\ 3\times 10\sin 600t+4\times 10^{-4}\times 6000\cos 600t$$

$$=3\sin 600t+2\ 4\cos 600t \quad . \quad . \quad . \quad .\text{(ii)}$$

Assume that (ii) may be written in the form

$$A\sin(600t+E)$$

This, on expansion, gives (p 27)

$$A\cos E\sin 600t+A\sin E\cos 600t \quad .. \quad \text{(iii)}$$

Hence, comparing (iii) with (ii),

$$A\cos E=3, \quad \text{and} \quad A\sin E=2\ 4$$

Squaring and adding,

$$A^2(\sin^2 E+\cos^2 E)=3^2+(2\ 4)^2=14\ 76;$$

$$A^2=14\ 76,$$

or

$$A=3\ 84$$

Also $\qquad E=\tan^{-1}0{\cdot}8=38°\ 39\ 5'$

Hence, $\qquad$ lowest value of $C=-10,$

$\qquad\qquad$,, $\quad$,, $\quad V=-3\ 84;$

$\qquad$ highest value of $C=10,$

$\qquad\qquad$,, $\quad$,, $\quad V=3\ 84$

Variation of atmospheric pressure with altitude.—If p_0 is the pressure, ρ_0 the density of the air at sea-level, and p the pressure, and ρ the density at a height h,

$$dp=-\rho dh$$

The negative sign indicates that the pressure decreases as the altitude increases Hence

$$\frac{dp}{dh}=-\rho \quad . \quad \text{(i)}$$

To express the density ρ in terms of the pressure and density ρ_0 at sea-level, we have, from Boyle's Law,

$$p\times\rho_0=\rho p_0,$$

or

$$\rho=\frac{p\times\rho_0}{p_0}$$

Substitute this value in (1), then

$$\frac{dp}{dh} = \frac{-p \times \rho_0}{p_0},$$

$$\frac{dp}{p} = \frac{-\rho_0}{p_0} dh,$$

$$\log_e p = \frac{-\rho_0}{p_0} h + \log_e c$$

Hence $$p = ce^{\frac{-\rho_0}{p_0} h}.$$

To obtain the value of the constant c we notice that at sea-level, where $h = 0$, $p = p_0$, $c = p_0$,

$$p = p_0 e^{\frac{-\rho_0}{p_0} h}$$

Differential Equations.—Type II $\frac{d^2 x}{dt^2} = -Fx$, where F is a constant

The solution is $x = A \sin\sqrt{F}t + B \cos\sqrt{F}t$, ... (1) where A and B are constants of integration. (The solution obtained may be readily verified by reversing the operation, i e finding $\frac{d^2 y}{dx^2}$ and comparing with x Also it contains two arbitrary constants as required by the theory—the equation being of the second order)

This is an important equation, and is a typical case of harmonic motion, occurring, for example, in the small oscillations of a spring or of a pendulum It is also used in the so-called Theory of Struts and may be written in the form

$$\frac{d^2 y}{dx^2} = -Cy.$$

The solution may be obtained from Eq (1) by writing x for t and C for F,

$$y = A \sin\sqrt{C}x + B \cos\sqrt{C}x.$$

If s denotes the space, or distance from some fixed point, and t the time in seconds, then the equation may be written

$$\frac{d^2 s}{dt^2} + q^2 s = 0$$

The solution would be

$$s = A \sin qt + B \cos qt$$

Ex 8 Let $A=0$, $B=7$, $q=3$ Then the equation becomes

$$s=7\cos 3t,$$

this is referred to on p 135

If the differential equation is

$$\frac{d^2s}{dt^2}-k^2s=0,$$

the solution is $$s=Ae^{-kt}+Be^{kt},$$

as may be proved by obtaining the second differential

A particular solution of this equation is given by

$$s=Ae^{-kt}$$

The reader should refer to pp 141, 145 in which are given figures of the curves

$$s=Ae^{\pm kt} \text{ and } y=Ae^{kx}\sin(bx+c)$$

Vibration of a bar or spring.—The deflection of a bar is proportional to the load, and a bar when loaded may be made to vibrate The periodic time is equal to $2\pi\sqrt{\frac{m}{F}}$, where m is the mass of the load, and F the force required to produce unit displacement

The periodic time T of a weight P of mass m suspended at one end of a spiral spring, the other end of which is fastened to a suitable support, is in like manner given by

$$T=2\pi\sqrt{\frac{m}{F}}$$

Let F denote the force required to produce unit displacement When P is displaced a distance x from its equilibrium position the resultant upward force is Fx The acceleration in a downward direction (*i e* in a direction tending to increase x) is $\frac{d^2x}{dt^2}$

The acceleration in the upward direction is $-\frac{d^2x}{dt^2}$

$$\text{As force} = \text{mass} \times \text{acceleration} = -\frac{md^2x}{dt^2},$$

where m is the mass of the body at P,

$$m\frac{d^2x}{dt^2}=-Fx \qquad \ldots \quad (1)$$

To solve this, suppose

$$x = A \sin pt + B \cos pt,$$

then
$$\frac{dx}{dt} = Ap \cos pt - Bp \sin pt,$$

and
$$\frac{d^2x}{dt^2} = -Ap^2 \sin pt - Bp^2 \cos pt$$
$$= -p^2 x \qquad \text{(ii)}$$

Now (i) can be written in the form

$$\frac{d^2x}{dt^2} = -\frac{F}{m}x \qquad \text{(iii)}$$

Hence, comparing (ii) and (iii),

$$p = \sqrt{\frac{F}{m}},$$

$$x = A \sin \sqrt{\frac{F}{m}}t + B \cos \sqrt{\frac{F}{m}}\,t = C \sin\left(\sqrt{\frac{F}{m}}\,t + a\right) \qquad \text{(iv)}$$

and
$$\frac{dx}{dt} = A\sqrt{\frac{F}{m}} \cos \sqrt{\frac{F}{m}}t - B\sqrt{\frac{F}{m}} \sin \sqrt{\frac{F}{m}}t \qquad \text{(v)}$$

Let the initial displacement of the spring be a, $\frac{dx}{dt}$ simply denotes the velocity of P, and, when the displacement is a, the velocity is zero, or P, at the instant considered, is at rest

Hence $x = a$ when $t = 0$

Also $\frac{dx}{dt} = 0$ when $t = 0$,

$$a = A \sin\left(\sqrt{\frac{F}{m}} \times 0\right) + B \cos\left(\sqrt{\frac{F}{m}}\,0\right) \text{ from (iv)},$$

or $a = B$

Also $0 = A\sqrt{\frac{F}{m}}$ from (v); $\quad A = 0$

Thus, we obtain

$$x = a \cos \sqrt{\frac{F}{m}}\,t \qquad \text{(vi)}$$

If the constant $\sqrt{\frac{F}{m}}$ be denoted by n, (vi) becomes $x = a \cos nt$.

Substituting various values for nt, we can obtain various data with regard to the motion, thus, when $nt = 0$, $x = a$

When $\quad nt=\frac{\pi}{2}, \quad x=0$; $\quad$ body is at P

" $\quad nt=\pi, \quad x=a(\cos\pi)=-a$

" $\quad nt=\frac{3\pi}{2}, \quad x=0.$

" $\quad nt=2\pi, \quad x=a$

Hence, as nt increases from 0 to 2π, the body moves through a complete cycle into the initial position;

$$T=2\pi\sqrt{\frac{m}{F}},$$

where T denotes the periodic time.

Similarly, the variations of the velocity can be traced by reference to the values of $\frac{dx}{dt}$

Ex 9 The result obtained for the periodic time can easily be verified by experiment When a load W of 10·5 lbs is suspended from a spiral spring it is found that 190 swings are made in one minute Also, 10 8 lbs is required to stretch the string through unit distance one inch ($g=32\ 2$ ft. per sec per sec)

As $\quad$ 1 minute $=$ 60 seconds,

$$T=\tfrac{60}{190}=\tfrac{6}{19}=0\ 3158 \text{ sec.}$$

Also, $\quad T=2\pi\sqrt{\frac{m}{F}},$

where $\quad m=10{\cdot}5\div g=10\ 5\div 32\ 2\times 12,$

and $\quad F=10\ 8$ lbs

(as the unit distance is 1 inch, $g=32{\cdot}2\times 12$ ins per sec. per sec.);

$$T=2\pi\sqrt{\frac{10\ 5}{32\ 2\times 12\times 10{\cdot}8}}=0{\cdot}3152 \text{ sec.}$$

It will be noticed that in the preceding solution the mass of the spring itself has not been taken into account; in fact we have made the assumption that the weight of the spring, and therefore its mass, is negligible in comparison with the vibrating mass at the end of the spring.

Allowance for the weight of the spring may be made by adding a fractional part of the mass of the spring to the

vibrating mass at the end of the spring The numerical value of this fractional part is readily obtained Thus, if ρ denotes the density of the material of the spring, and if v denotes the velocity of the vibrating spring at a distance x from the point of support, we obtain

$$\frac{\rho}{2}\frac{x}{l}v;$$

$$\therefore \text{ kinetic energy} = \tfrac{1}{2}mv^2 + \int_0^l \frac{\rho}{2}\left(\frac{x}{l}v\right)^2 dx = \tfrac{1}{2}mv^2 + \left[\frac{\rho}{2}\frac{v^2}{l^2}\frac{x^3}{3}\right]_0^l$$

$$= \tfrac{1}{2}v^2\left(m + \frac{\rho l}{3}\right)$$

Hence, the mass of the spring may be taken into account by adding **one-third its mass** to the mass at the end of the spring

Ex 10 A spiral spring is supported at the upper end, and when a weight of 7 lbs is hung on to the lower end, an extension of 0 1 foot is produced

Find the time of a vertical oscillation (1) neglecting the mass of the spring, (2) supposing the spring weighs 0 6 lb, and a proper allowance for its mass is added to the 7 lb weight.

(1) In the formula for the periodic time,

$$t = 2\pi\sqrt{\frac{m}{F}},$$

$$m = \frac{7}{32\ 2} \text{ and } F = 7\times 10 = 70 \text{ lbs };$$

$$t = 2\pi\sqrt{\frac{7}{32\ 2\times 70}} = 2\pi\sqrt{\frac{1}{322}}$$

$$\log t = \log 2 + \log\pi - \tfrac{1}{2}\log 322 = \bar{1}\ 5443;$$

$$t = 0\ 3501 \text{ sec}$$

(2) Adding $\frac{1}{3}$ the mass of the spring,

$$M = \frac{7}{32{\cdot}2} + \frac{0\ 2}{32\ 2} = \frac{7\ 2}{32{\cdot}2}, \text{ also } F = 70 \text{ lbs.}$$

$$\therefore t = 2\pi\sqrt{\frac{7{\cdot}2}{32\ 2\times 70}} = 0\ 355 \text{ sec}$$

Vibration of a beam or rod.—If a bar or rod is supported at its ends A and B and loaded at the centre with a load W, the deflection δ will be given by the formula

$$\delta = \frac{Wl^3}{48EI}, \qquad (1)$$

where l is the length between supports, E is the modulus of elasticity of the material, and I the moment of inertia The length and deflection may be expressed in centimetres, in inches, or in feet, W, E, and I must obviously be in the same units Expressing l and δ in inches, then E will be expressed in pounds per square inch and I in inch units.

If the value of E for a given material is known, from Eq (1) the numerical value of δ for a given weight W can be calculated Or, conversely, if δ is carefully measured from experiments, then E can be obtained

When a bar supported at the ends and loaded at its middle point with a weight W, is made to vibrate, the periodic time of a vibration can be calculated from the formula $T = 2\pi\sqrt{\frac{M}{F}}$, where F is the force necessary to produce unit deflection.

A verification of the result may easily be obtained by experiment

Ex. 11 A wooden rectangular beam or rod rests in a horizontal position on two knife edges 36 inches apart Find the periodic time of a vibration and the number of vibrations per second when the load at the centre is 10 lbs (Given $E = 1\ 865 \times 10^6$, depth of rod $\frac{1}{2}$ inch, width 1 inch)

The value of I for a rectangle of sides b and d is $\frac{1}{12}bd^3$ (p 432)

Substituting in (1), $\qquad \delta = \frac{Wl^3}{4Ebd^3}; \qquad$ (ii)

$$W = \frac{4 \times 1\ 1865 \times 10^6 \times 1 \times (0{\cdot}5)^3 \times 12}{36^3} \times \delta_1$$

Obviously, δ_1 must denote the deflection in feet when $g = 32\ 2$, and in inches when $g = 32\ 2 \times 12$

W, the load required to produce unit deflection of 1 foot, is found to be 240 lbs ; the weight of the rod is 8 5 oz , and for the purpose of this calculation $\frac{5}{8}$ths of this may be assumed to act at its middle point ;

$$\therefore \text{ mass} = \frac{10}{32 \cdot 2} + \frac{\frac{3}{8} \times 8\ 5}{16 \times 32 \cdot 2} = 0\ 3725\,;$$

$$\text{periodic time } T = 2\pi\sqrt{\frac{0\ 3725}{240}} = 0\ 2475 \text{ second,}$$

or 4 04 vibrations per second

The formula used may be readily proved by the student in a manner similar to that used in finding the time of a vibration of a mass suspended from a spring (p 481) In fact a beam or rod loaded in the manner indicated is only one form of spring.

Simple pendulum.—The nearest approximation to a so-called simple pendulum consists of a small heavy body, such as a leaden bullet, at one end of a fine string, the other end of the string being fixed to a suitable support and the pendulum made to perform small oscillations in a vertical plane When the arc of vibration is small, the time of vibration may be obtained in a very simple manner as follows

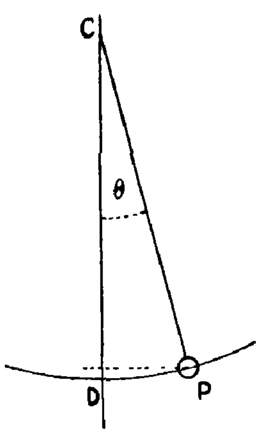

FIG. 178 —Simple pendulum

Let P (Fig 178) denote a small mass at one end of a string of length l, the other end of which is fastened to a fixed support C

Let m denote the mass of the particle at P, and θ the angle DCP

The two components of the force mg, one along the string PC, the other at right angles to it, may be obtained The former component, $mg \cos\theta$, produces tension in the string, the latter, $mg \sin\theta$, produces the acceleration of P.

From the relation, force = mass × acceleration

$$\text{acceleration of } P = \frac{mg \sin\theta}{m} = g \sin\theta.$$

The relation between acceleration and displacement in S.H.M. is furnished by

$$\frac{\text{acceleration}}{\text{displacement}}=\omega^2=\frac{2^2\pi^2}{T^2}\text{ (p. 135)},$$

$$\frac{2^2\pi^2}{T^2}=\frac{g\sin\theta}{l\theta}$$

As the angle is supposed to be small, the sine of the angle is very approximately equal to its circular measure (p. 383)

Hence we obtain $$\frac{2^2\pi^2}{T^2}=\frac{g}{l},$$

$$T=2\pi\sqrt{\frac{l}{g}}, \qquad (1)$$

where T denotes the periodic time of a vibration

In the preceding case the arc of swing has been assumed to be very small, when this is not the case, Eq (1) cannot be used to find the periodic time

The relation between force and acceleration is

force = mass × acceleration,

or torque = (moment of inertia) × (angular acceleration),

the former being expressed in linear, the latter in angular, motion.

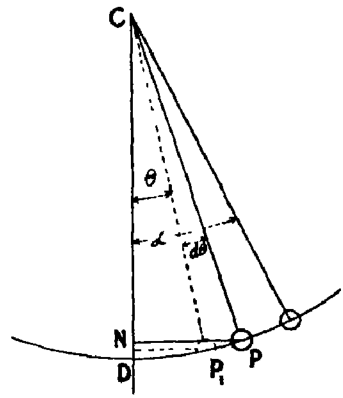

FIG 179

Let m be the mass at P (Fig 179), l the length of CP, P_1 and P two positions of P, the angle $DCP_1=\theta$, and $P_1CP=d\theta$ Draw PN perpendicular to DC

Then, torque $= m\times PN = ml\sin\theta$

Also, moment of inertia of P about C is ml^2,

$$mgl\sin\theta=-ml^2\frac{d^2\theta}{dt^2}.$$

The negative sign denotes that θ is decreasing; dividing by ml^2, we obtain

$$\frac{d^2\theta}{dt^2}+\frac{g\sin\theta}{l}=0. \qquad \text{...... .. . ...(1)}$$

Multiply by $2\frac{d\theta}{dt}$, and integrate between limits $\left(\frac{d\theta}{dt}=0\right.$, when $\theta=\alpha$, where α is the greatest value of $\left.\theta\right)$,

$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}}(\cos\theta-\cos\alpha)^{\frac{1}{2}},$$

and
$$t=\sqrt{\frac{l}{2g}}\int\frac{d\theta}{(\cos\theta-\cos\alpha)^{\frac{1}{2}}}$$

As $\cos\theta=1-2\sin^2\frac{\theta}{2}$ and $\cos\alpha=1-2\sin^2\frac{\alpha}{2}$, the periodic time becomes

$$T=\frac{4}{2}\sqrt{\frac{l}{g}}\int_0^\alpha\frac{d\theta}{\sqrt{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)}}$$

Since α is the greatest value of θ, we may assume

$$\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\sin\phi$$

And, since when $\theta=\alpha$, $\sin\phi=1$ or $\frac{\pi}{2}$, and when $\theta=0$, $\sin\phi=0$ or $\phi=0$, the limits of integration are $\frac{\pi}{2}$ and 0

Then $\frac{1}{2}\cos\frac{\theta}{2}d\theta=\sin\frac{\alpha}{2}\cos\phi d\phi$,

$$T=2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{2d\phi}{\left(1-\sin^2\frac{\alpha}{2}\sin^2\phi\right)^{\frac{1}{2}}} \qquad \text{(ii)}$$

Expand the fraction in (ii) by the Binomial Theorem,

$$T=4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\left\{1+\tfrac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi+\quad\right\}d\phi$$

$$=4\sqrt{\frac{l}{g}}\left[\phi+\tfrac{1}{2}\sin^2\frac{\alpha}{2}\left\{\frac{\phi}{2}-\frac{\sin 2\phi}{4}\right\}+\text{etc}\right]_0^{\frac{\pi}{2}}$$

$$=4\pi\sqrt{\frac{l}{g}}\left(\frac{1}{2}+\frac{1}{8}\sin^2\frac{\alpha}{2}\right)$$

+ terms which may be neglected

$$=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\alpha^2}{16}\right), \text{ approx}$$

If θ is small, θ may be written for $\sin\theta$ in Eq. (i), and the formula for a simple pendulum obtained.

Ex 12. If l is the length of a seconds pendulum, find the number of seconds lost in a day when the arc of vibration is 9°

We may denote by T the periodic time of a seconds pendulum, and by T' that of a pendulum which swings through an angle of 9° on each side of the vertical

As 24 hours is 24×3600 seconds,

$$\therefore \text{ loss in seconds} = 24 \times 3600\, T\left(\frac{1}{T} - \frac{1}{T'}\right)$$

$$= 24 \times 3600\left(1 - \frac{T}{T'}\right)$$

$$= 24 \times 3600\left(1 - \frac{1}{1 + \frac{\alpha^2}{16}}\right)$$

$$= 24 \times 3600\left(\frac{\alpha^2}{16 + \alpha^2}\right)$$

$$= \frac{24 \times 3600(0\ 1571)^2}{16 + (0\ 1571)^2} = 133\ 1 \text{ secs}$$

Ex 13 A uniform straight plank rests with its middle point upon a rough horizontal cylinder, the axes of the cylinder and plank being perpendicular to each other Supposing the plank to be slightly displaced so as to remain always in contact with the cylinder without sliding determine the periodic time

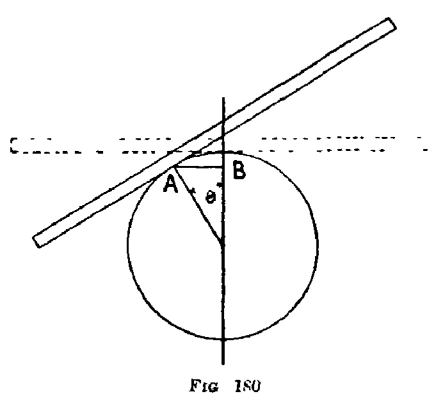

FIG 180

Let $2l$ denote the length of the plank and r the radius of the cylinder, and let m denote the mass of the plank

Assume the plank to be displaced through a small angle θ so that the plank and cylinder are in contact at a point A (Fig 180) Draw AB perpendicular to the vertical line passing through the centre of the cylinder,

then moment of restoring force is $mg \times AB = mgr \sin\theta$ (very approximately);

$$mgr \sin\theta = -I\frac{d^2\theta}{dt^2} \qquad . \qquad (1)$$

The value of I for a thin rod, length $2l$, about an axis passing perpendicularly through its middle point is $\frac{ml^2}{3}$ (p 431)

Hence, substituting in (1),

$$mgr \sin\theta + \frac{ml^2}{3}\frac{d^2\theta}{dt^2} = 0$$

As the angle is small, $\sin\theta$ is approximately equal to θ;

$$gr\theta + \frac{l^2}{3}\frac{d^2\theta}{dt^2} = 0.$$

Solving as in Type II (p 480),

$$\theta = A \sin\sqrt{\frac{3gr}{l^2}}\,t + B\cos\sqrt{\frac{3gr}{l^2}}\,t,$$

$$\text{periodic time} = \frac{2\pi l}{\sqrt{3gr}}$$

Vibration of an indicator — In some cases, such as, for example, in a steam engine indicator, the calculation for the frequency of a vibration must include the consideration of two or more vibrating masses Thus, in Fig 181, pressure on the piston P compresses a spring S, the motion of the piston rod, by means of suitable links, gives motion to a lever centred at A The other end C, carrying a pencil point, indicates on an enlarged scale the motion of the piston P.

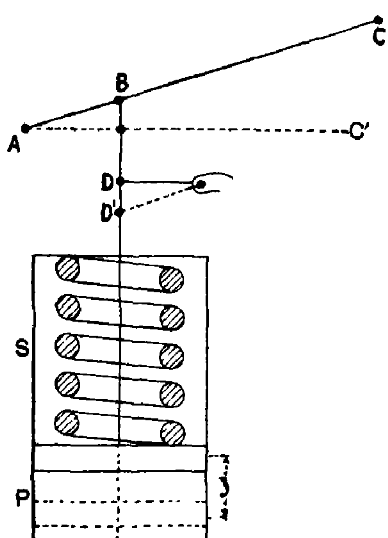

FIG 181

The frequency may be calculated by estimating the masses of the moving parts and the shortening produced in the spring by a given pressure Let M denote the mass in pounds of piston

and rod including the link BD and one-third the mass of the spring Let I denote the moment of inertia (in ft. lbs units) of the lever ABC The initial position of the lever is at AC'. When the piston moves through a distance y, the position of the lever may be denoted by the line AC making an angle θ with AC'.

If c denote the compression (in feet) of the spring per pound of load, and a in the same units the initial compression of the spring when the lever is horizontal

Let F be the compressive force, then $\frac{d^2\theta}{dt^2}=\frac{AB\ F}{I}$, where I denotes the moment of inertia about A

$$F=\frac{I}{AB}\frac{d^2\theta}{dt^2}=\frac{I}{AB^2}\frac{d^2y}{dt^2},$$

if θ is so small that $y=\theta\times AB$, then, from the relation

mass × acceleration = force acting,

we obtain the equation

$$M\frac{d^2y}{dt^2}+\frac{I}{AB^2}\frac{d^2y}{dt^2}=-F\frac{y+a}{a}.$$

This gives for the periodic time

$$T=2\pi\sqrt{\frac{\left(M+\frac{I}{AB^2}\right)a}{F}}$$

$$=\frac{2\pi}{AB}\sqrt{\frac{(M\times AB^2+I)a}{F}}$$

It will be noticed that the mass of the lever and its length are taken into account in the moment of inertia

Struts.—A rod of length $2l$ acted on by compressive forces in the direction of its length (Fig 182) is called a **strut.**

The equation connecting the force F, the deflection y, and the curvature is expressed by

$$\frac{Fy}{EI}=-\frac{d^2y}{dx^2}. \quad \ldots \quad (\text{i})$$

Let $n^2=\frac{F}{EI}$, then, as in the preceding case, (i) may be written

$$\frac{d^2y}{dx^2}=-n^2y.$$

Hence $$y=A\sin nx+B\cos nx \quad \ldots\ldots \quad (\text{ii})$$

From (ii), by differentiation,

$$\frac{dy}{dx}=An\cos nx-Bn\cos nx \quad . \qquad . \qquad \text{(iii)}$$

Now the tangent to the curve is parallel to the axis of x at O, where $x=0$, and at the two ends M and N, where $x=l$ and $x=-l$, respectively, $y=0$

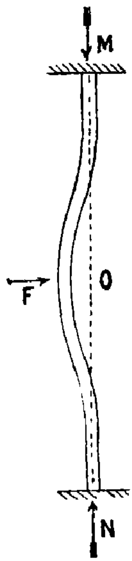

FIG 182

Putting $\frac{dy}{dx}=0$ and $x=0$, in (iii),

$$0=An\,, \qquad A=0$$

Hence, substituting in (ii),

$$y=B\cos nx=B\cos\sqrt{\frac{F}{EI}}x \qquad \text{(iv)}$$

When $x=0$, $y=B$ Hence, the constant B denotes the maximum deflection, that is, the deflection of the strut in the centre

Again, when $x=l$ or $-l$, $y=0$ Hence, from (iv),

$$0=B\cos\sqrt{\frac{F}{EI}}l \qquad . \qquad \text{(v)}$$

It follows at once from Eq (v) either that $B=0$ or

$$\cos\left(\sqrt{\frac{F}{EI}}l\right)=0$$

Hence, $\cos\left(\sqrt{\frac{F}{EI}}l\right)$ must be 0, since, from the above considerations, B is not zero, hence the angle must be $\frac{\pi}{2}$, $\frac{3\pi}{2}$, or other odd multiple of $\frac{\pi}{2}$,

$$\sqrt{\frac{F}{EI}}l=\frac{\pi}{2},$$

or

$$F=\frac{EI\pi^2}{4l^2}$$

Ends fixed.—The maximum value of F_1 when the ends of a strut are fixed may be obtained as follows

From (ii),

$$y=A\sin nx+B\cos nx$$

$$=A\sin\sqrt{\left(\frac{F_1}{EI}\right)}x+B\cos\sqrt{\left(\frac{F_1}{EI}\right)}x\,,$$

$$\frac{dy}{dx}=\sqrt{\frac{F_1}{EI}}A\cos\sqrt{\frac{F_1}{EI}}x-B\sqrt{\frac{F_1}{EI}}\sin\sqrt{\frac{F_1}{EI}}x$$

In this case $\frac{dy}{dx}=0$ when $x=0$, also when $x=l$ and when $x=-l$

Let $x=0$, then

$$0=A\sqrt{\frac{F_1}{EI}}, \qquad A=0, \qquad y=B\cos\sqrt{\frac{F_1}{EI}}x$$

Differentiating,

$$\frac{dy}{dx}=-B\sqrt{\frac{F_1}{EI}}\sin\sqrt{\frac{F_1}{EI}}x$$

Now, when $x=l$, $\frac{dy}{dx}=0$,

$$0=-B\sqrt{\frac{F_1}{EI}}\sin\sqrt{\frac{F_1}{EI}}l$$

Hence, either $\qquad B=0$ or $\sin\sqrt{\frac{F_1}{EI}}l=0$

Therefore, as B cannot be 0, the angle must be π or 2π, etc

Taking the smallest value, we have

$$\sqrt{\frac{F_1}{EI}}l=\pi\,;$$

$$F_1=\frac{EI\pi^2}{l^2}$$

The formulae for F and F_1 are known as **Euler's** formulae.

Hence, a strut fixed in direction at both ends is four times as strong as a strut in which one end is not fixed in direction

Ex. **14.** Find the breaking load of a wrought-iron cylindrical pillar or strut, 3 inches diameter and 6 feet long $E=29\times10^6$

Here $I=\frac{\pi r^4}{4}=\frac{\pi\times3^4}{4^3}$; $l=6\times12$;

$$F=\frac{29\times10^6\times\pi^3\times3^4}{4^3\times6^2\times12^2}=\frac{29\times10^6\times\pi^3\times3^4}{4^3\times72^2}\text{ lbs}$$

$$\log F=\log29+6\log10+3\log\pi+4\log3-(3\log4+2\log72+\log2240)\,;$$

$$F=98\text{ tons, approx.}$$

Differential Equations: Type III.—The differential equation given by Type II. (p 480) is of great utility and importance, and is that arrived at in very many problems on vibration. A more general form is, however, sometimes wanted, as in the case of damped vibrations (p 142), and the equation may be written in the form

$$\frac{d^2s}{dt^2}+2F\frac{ds}{dt}+k^2s=0$$

We may surmise that $s=Ae^{at}$ will be a solution. Trying this value, we obtain

$$\frac{ds}{dt}=Aae^{at}$$

and $$\frac{d^2s}{dt^2}=Aa^2e^{at},$$

$$Aa^2e^{at}+Aae^{at}(2F)+k^2Ae^{at}=0,$$

or $$s(a^2+2Fa+k^2)=0$$

Hence, we see that if a satisfies this equation, s will satisfy the given differential equation.

Solving the quadratic

$$a=-F\pm\sqrt{F^2-k^2}$$

If F is $>k$ the values of a are both real, and the solution is

$$s=Ae^{(-F+\sqrt{F^2-k^2})t}+Be^{(-F-\sqrt{F^2-k^2})t}.$$

If F is equal to k, the values of a are equal, and we find

$$s=(A+Bt)e^{-Ft}$$

as the solution.

Substituting this value in the differential equation, we obtain, since $k=F$,

$$\frac{ds}{dt}=-Fe^{-Ft}(A+Bt)+Be^{-Ft}$$

and $$\frac{d^2s}{dt^2}=+F^2e^{-Ft}(A+Bt)-2BFe^{-Ft};$$

$$\frac{d^2s}{dt^2}+2F\frac{ds}{dt}+F^2s=0$$

becomes $$F^2e^{-Ft}(A+Bt)-2BFe^{-Ft}$$

$$-2F^2e^{-Ft}(A+Bt)+2BFe^{-Ft}+F^2e^{-Ft}(A+Bt),$$

which is identically zero.

If F is less than k, α becomes partly imaginary; the solution may, however, be re-written, and we find

$$s = e^{-Ft}\{A \sin \sqrt{(k^2 - F^2)}\,t + B \cos \sqrt{(k^2 - F^2)}\,t\}.$$

This result may be proved, by trial, to satisfy the equation as before

If F is zero the solution becomes

$$s = A \sin kt + B \cos kt,$$

and the differential equation is

$$\frac{d^2s}{dt^2} + k^2 s = 0,$$

i e the equation of Type II.

As the differential equation contains $\frac{d^2s}{dt^2}$, not more than two arbitrary constants must have disappeared, a solution therefore containing only one would be incomplete, and probably in an actual case would not be sufficient to solve the problem For more detailed information the reader should consult works on differential equations A few simple examples are given.

Ex 15 Solve the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

Put $y = Ae^{\alpha x}$, and we obtain

$$\alpha^2 + 3\alpha + 2 = 0,$$

$$\text{or } \alpha = -1 \text{ and } \alpha = -2.$$

Solution is $\quad y = Ae^{-x} + Be^{-2x}$

Ex 16 Solve the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

Here the roots are equal;

$$y = (A + Bx)e^{-2x}.$$

Ex 17 $$\frac{1}{4}\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{2} = 0.$$

Substituting, we find

$$\alpha^2 + 4\alpha + 2 = 0;$$

$$y = e^{-2x}\{A \sin \sqrt{2}x + B \cos \sqrt{2}x\}.$$

MISCELLANEOUS EXERCISES. XLV

Solve the equations:

1. $\dfrac{d^2y}{dx^2}+7\dfrac{dy}{dx}+9y=0.$ 2. $3\dfrac{d^2y}{dx^2}+25\dfrac{dy}{dx}-18y=0$

3. $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+9y=0.$ 4. $2\dfrac{d^2y}{dx^2}+17\dfrac{dy}{dx}+40y=0$

5. In how many years will a sum of money quadruple itself at 5 per cent per annum?

6. A wet rope touches half way round a rough cylindrical post, and the rope begins to slip when the tensions at the two ends of the rope are 7 lbs. and 56 lbs respectively. Find the coefficient of friction between the rope and the cylinder Also find approximately the weight which would be supported if the rope were to make an additional complete turn

7. Write down the relation between the tensions T_1 and T_2 when a belt or rope is just about to slip on a pulley If T_1 may be three times T_2 when the angle is 180° without making the rope slip, what will the ratio be when the rope makes a complete turn?

8. If a string, hanging in a vertical plane over a rough horizontal cylinder with 20 lbs hanging at one end and 2 lbs at the other, be on the point of slipping, find the coefficient of friction between the cylinder and the string

9. The slope of a curve at a point whose abscissa is x is given by x^2-x+1 Given that the curve passes through the point $x=1$, $y=2$; find the equation to the curve Also find the value of y when $x=3$

10. At what point on the curve $y=2x^3$ is the tangent parallel to the line which touches the curve $y=3x^2-6x+2$ at the point P whose abscissa x is 1·4? Also find the radius of curvature at P.

11. Find the points of intersection of the curves

$$y^2=4ax \text{ and } y^2=\frac{4}{27a}(x-2a)^3$$

12. Divide 30 into two parts such that the square of the first together with twice the square of the second shall be a minimum

13. Given the three points (0, 0) (2, 8) (4, 20) Assuming the equation of the curve passing through the three points to be $y=a+bx+cx^2$, find the area between the axis of x and the ordinates $x=0$, $x=4$. If the curve rotates about the axis of x, find the volume.

14. Find the volume of the segment of a sphere the height of the segment being one-half the radius

15. Draw the graph of $y=\frac{1}{2}(e^x+e^{-x})$ Find the area bounded by the curve and the two ordinates where $x=0$, $x=1·5$. If this area rotates about the axis of x; find the volume described.

Table I.

USEFUL NUMBERS AND FORMULAE

$\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$, $\sqrt{5}=2{\cdot}236$, $\sqrt{6}=2{\cdot}449$.

$\pi=3{\cdot}1416$ or $3{\cdot}142=\frac{22}{7}$ $\frac{1}{\pi}=0{\cdot}3183$

$\pi^2=9{\cdot}872$ 1 inch $=2{\cdot}54$ cm

1 lb. $=453{\cdot}6$ grams, $2\frac{1}{5}$ lbs. $=1$ kilogram

1 gallon of water $=10$ lbs. $=0{\cdot}1605$ cub. ft.

1 cubic foot of water $=62{\cdot}3$ lbs.

Volts $\times$ ampères $=$ watts

1 horse-power $=33000$ ft. lbs. per min.
$=746$ watts

1 radian $=57{\cdot}3$ degrees

$\log \pi = 0{\cdot}4972$

$\log 2{\cdot}718 = 0{\cdot}4343$

$\log {\cdot}7854 = \bar{1}{\cdot}8951$

$\log 62{\cdot}3 = 1{\cdot}7945$

$\log 1728 = 3{\cdot}2375$

$\log {\cdot}5236 = \bar{1}{\cdot}7190$

$\log {\cdot}1605 = \bar{1}{\cdot}2054$

To convert common into Napierian logarithms, multiply by $2{\cdot}3026$ ($e=2{\cdot}718$).

Mensuration Formulae. In the following formulae A denotes area; S, surface; V, volume; a, b, c, the sides of a figure; h, the altitude; l, the slant height; R and r, radii of circles.

Rectangle *or Parallelogram.* $A=ah$.

Triangle. $A=\frac{1}{2}ah$, or $\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$.

Trapezium. Parallel sides a and b. $A=\frac{1}{2}(a+b)h$.

Circle. Circumference $=2\pi r$, $A=\pi r^2$ or $\pi(R^2-r^2)$.

Ellipse. Semi-axes a and b. $A=\pi ab$.

Simpson's Rule. $A=\frac{s}{3}(A_1+4B+2C)$ where s is the space or distance between two consecutive ordinates. A_1 is the sum of first and last ordinates, B is sum of even, and C is sum of the odd ordinates.

Prism. $S=2(ab+bc+ac)$, $V=abc$, diagonal $=\sqrt{a^2+b^2+c^2}$.

Cylinder. $S=2\pi rh+2\pi r^2$, $V=\pi r^2 h$.

Cone. $S=\pi rl+\pi r^2$, $V=\frac{1}{3}\pi r^2 h$.

Sphere. $S=4\pi r^2$, $V=\frac{4}{3}\pi r^3=0{\cdot}5236d^3$.

Ring. $S=4\pi^2 Rr$, $V=2\pi^2 r^2 R$.

Weight in lbs. per cub. in.—Cast iron, 0·26; Wrought iron, 0·28; Steel, 0·29; Brass, 0·298; Copper, 0·319; Lead, 0·414.

Table II.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Table II.

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Table III.

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Table III.

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Table IV.

NATURAL SINES

	0′	10′	20′	30′	40′	50′	1	2	3	4	5	6	7	8	9
0	0000	0029	0058	0087	0116	0145	3	6	9	12	15	17	20	23	26
1	0175	0204	0233	0262	0291	0320	3	6	9	12	15	17	20	23	26
2	0349	0378	0407	0436	0465	0494	3	6	9	12	15	17	20	23	26
3	0523	0552	0581	0610	0640	0669	3	6	9	12	15	17	20	23	26
4	0698	0727	0756	0785	0814	0843	3	6	9	12	15	17	20	23	26
5	0872	0901	0929	0958	0987	1016	3	6	9	12	14	17	20	23	26
6	1045	1074	1103	1132	1161	1190	3	6	9	12	14	17	20	23	26
7	1219	1248	1276	1305	1334	1363	3	6	9	12	14	17	20	23	26
8	1392	1421	1449	1478	1507	1536	3	6	9	12	14	17	20	23	26
9	1564	1593	1622	1650	1679	1708	3	6	9	12	14	17	20	23	26
10	1736	1765	1794	1822	1851	1880	3	6	9	12	14	17	20	23	26
11	1908	1937	1965	1994	2022	2051	3	6	9	11	14	17	20	23	26
12	2079	2108	2136	2164	2193	2221	3	6	9	11	14	17	20	23	26
13	2250	2278	2306	2334	2363	2391	3	6	8	11	14	17	20	23	25
14	2419	2447	2476	2504	2532	2560	3	6	8	11	14	17	20	23	25
15	2588	2616	2644	2672	2700	2728	3	6	8	11	14	17	20	22	25
16	2756	2784	2812	2840	2868	2896	3	6	8	11	14	17	20	22	25
17	2924	2952	2979	3007	3035	3062	3	6	8	11	14	17	19	22	25
18	3090	3118	3145	3173	3201	3228	3	6	8	11	14	17	19	22	25
19	3256	3283	3311	3338	3365	3393	3	5	8	11	14	16	19	22	25
20	3420	3448	3475	3502	3529	3557	3	5	8	11	14	16	19	22	25
21	3584	3611	3638	3665	3692	3719	3	5	8	11	14	16	19	22	24
22	3746	3773	3800	3827	3854	3881	3	5	8	11	14	16	19	21	24
23	3907	3934	3961	3987	4014	4041	3	5	8	11	14	16	19	21	24
24	4067	4094	4120	4147	4173	4200	3	5	8	11	13	16	19	21	24
25	4226	4253	4279	4305	4331	4358	3	5	8	11	13	16	18	21	24
26	4384	4410	4436	4462	4488	4514	3	5	8	10	13	16	18	21	23
27	4540	4566	4592	4617	4643	4669	3	5	8	10	13	15	18	21	23
28	4695	4720	4746	4772	4797	4823	3	5	8	10	13	15	18	20	23
29	4848	4874	4899	4924	4950	4975	3	5	8	10	13	15	18	20	23
30	5000	5025	5050	5075	5100	5125	3	5	8	10	13	15	18	20	23
31	5150	5175	5200	5225	5250	5275	2	5	7	10	12	15	17	20	22
32	5299	5324	5348	5373	5398	5422	2	5	7	10	12	15	17	20	22
33	5446	5471	5495	5519	5544	5568	2	5	7	10	12	15	17	19	22
34	5592	5616	5640	5664	5688	5712	2	5	7	10	12	14	17	19	22
35	5736	5760	5783	5807	5831	5854	2	5	7	10	12	14	17	19	21
36	5878	5901	5925	5948	5972	5995	2	5	7	9	12	14	16	19	21
37	6018	6041	6065	6088	6111	6134	2	5	7	9	12	14	16	18	21
38	6157	6180	6202	6225	6248	6271	2	5	7	9	11	14	16	18	20
39	6293	6316	6338	6361	6383	6406	2	4	7	9	11	13	16	18	20
40	6428	6450	6472	6494	6517	6539	2	4	7	9	11	13	15	18	20
41	6561	6583	6604	6626	6648	6670	2	4	7	9	11	13	15	17	20
42	6691	6713	6734	6756	6777	6799	2	4	6	9	11	13	15	17	19
43	6820	6841	6862	6884	6905	6926	2	4	6	8	11	13	15	17	19
44	6947	6967	6988	7009	7030	7050	2	4	6	8	10	12	15	17	19

Table IV.

NATURAL SINES

	0′	10′	20′	30′	40′	50′	1	2	3	4	5	6	7	8	9
45	7071	7092	7112	7133	7153	7173	2	4	6	8	10	12	14	16	18
46	7193	7214	7234	7254	7274	7294	2	4	6	8	10	12	14	16	18
47	7314	7333	7353	7373	7392	7412	2	4	6	8	10	12	14	16	18
48	7431	7451	7470	7490	7509	7528	2	4	6	8	10	12	13	15	17
49	7547	7566	7585	7604	7623	7642	2	4	6	8	9	11	13	15	17
50	7660	7679	7698	7716	7735	7753	2	4	6	7	9	11	13	15	17
51	7771	7790	7808	7826	7844	7862	2	4	5	7	9	11	13	14	16
52	7880	7898	7916	7934	7951	7969	2	4	5	7	9	11	12	14	16
53	7986	8004	8021	8039	8056	8073	2	3	5	7	9	10	12	14	16
54	8090	8107	8124	8141	8158	8175	2	3	5	7	8	10	12	14	15
55	8192	8208	8225	8241	8258	8274	2	3	5	7	8	10	12	13	15
56	8290	8307	8323	8339	8355	8371	2	3	5	6	8	10	11	13	14
57	8387	8403	8418	8434	8450	8465	2	3	5	6	8	9	11	12	14
58	8480	8496	8511	8526	8542	8557	2	3	5	6	8	9	11	12	14
59	8572	8587	8601	8616	8631	8646	1	3	4	6	7	9	10	12	13
60	8660	8675	8689	8704	8718	8732	1	3	4	6	7	9	10	11	13
61	8746	8760	8774	8788	8802	8816	1	3	4	6	7	8	10	11	12
62	8829	8843	8857	8870	8884	8897	1	3	4	5	7	8	9	11	12
63	8910	8923	8936	8949	8962	8975	1	3	4	5	6	8	9	10	12
64	8988	9001	9013	9026	9038	9051	1	3	4	5	6	8	9	10	11
65	9063	9075	9088	9100	9112	9124	1	2	4	5	6	7	8	10	11
66	9135	9147	9159	9171	9182	9194	1	2	3	5	6	7	8	9	10
67	9205	9216	9228	9239	9250	9261	1	2	3	4	6	7	8	9	10
68	9272	9283	9293	9304	9315	9325	1	2	3	4	5	6	7	9	10
69	9336	9346	9356	9367	9377	9387	1	2	3	4	5	6	7	8	9
70	9397	9407	9417	9426	9436	9446	1	2	3	4	5	6	7	8	9
71	9455	9465	9474	9483	9492	9502	1	2	3	4	5	6	6	7	8
72	9511	9520	9528	9537	9546	9555	1	2	3	4	4	5	6	7	8
73	9563	9572	9580	9588	9596	9605	1	2	2	3	4	5	6	7	7
74	9613	9621	9628	9636	9644	9652	1	2	2	3	4	5	5	6	7
75	9659	9667	9674	9681	9689	9696	1	1	2	3	4	4	5	6	7
76	9703	9710	9717	9724	9730	9737	1	1	2	3	3	4	5	5	6
77	9744	9750	9757	9763	9769	9775	1	1	2	3	3	4	4	5	6
78	9781	9787	9793	9799	9805	9811	1	1	2	2	3	3	4	5	5
79	9816	9822	9827	9833	9838	9843	1	1	2	2	3	3	4	4	5
80	9848	9853	9858	9863	9868	9872	0	1	1	2	2	3	3	4	4
81	9877	9881	9886	9890	9894	9899	0	1	1	2	2	3	3	3	4
82	9903	9907	9911	9914	9918	9922	0	1	1	2	2	2	3	3	3
83	9925	9929	9932	9936	9939	9942	0	1	1	1	2	2	3	3	3
84	9945	9948	9951	9954	9957	9959	0	1	1	1	1	2	2	2	2
85	9962	9964	9967	9969	9971	9974	0	0	1	1	1	1	2	2	2
86	9976	9978	9980	9981	9983	9985	0	0	1	1	1	1	1	1	2
87	9986	9988	9989	9990	9992	9993	0	0	0	1	1	1	1	1	1
88	9994	9995	9996	9997	9997	9998	0	0	0	0	0	0	1	1	1
89	9998	9999	9999	1 0000	1 0000	1 0000	0	0	0	0	0	0	0	0	0

Table V.

NATURAL COSINES

Deg	0′	10′	20′	30′	40′	50′	1	2	3	4	5	6	7	8	9
0	1·0000	1·0000	1·0000	1·0000	9999	9999	0	0	0	0	0	0	0	0	0
1	9998	9998	9997	9997	9996	9995	0	0	0	0	0	0	0	0	0
2	9994	9993	9992	9990	9989	9988	0	0	0	0	0	0	1	1	1
3	9986	9985	9983	9981	9980	9978	0	0	1	1	1	1	1	1	1
4	9976	9974	9971	9969	9967	9964	0	0	1	1	1	1	1	1	2
5	9962	9959	9957	9954	9951	9948	0	1	1	1	1	1	2	2	2
6	9945	9942	9939	9936	9932	9929	0	1	1	1	2	2	2	2	2
7	9925	9922	9918	9914	9911	9907	0	1	1	2	2	2	3	3	3
8	9903	9899	9894	9890	9886	9881	0	1	1	2	2	2	3	3	3
9	9877	9872	9868	9863	9858	9853	0	1	1	2	2	3	3	3	4
10	9848	9843	9838	9833	9827	9822	1	1	2	2	2	3	3	4	4
11	9816	9811	9805	9799	9793	9787	1	1	2	2	3	3	4	4	5
12	9781	9775	9769	9763	9757	9750	1	1	2	2	3	3	4	5	5
13	9744	9737	9730	9724	9717	9710	1	1	2	3	3	4	4	5	6
14	9703	9696	9689	9681	9674	9667	1	1	2	3	4	4	5	5	6
15	9659	9652	9644	9636	9628	9621	1	2	2	3	4	4	5	6	7
16	9613	9605	9596	9588	9580	9572	1	2	2	3	4	5	5	6	7
17	9563	9555	9546	9537	9528	9520	1	2	3	3	4	5	6	7	7
18	9511	9502	9492	9483	9474	9465	1	2	3	4	4	5	6	7	8
19	9455	9446	9436	9426	9417	9407	1	2	3	4	5	6	6	7	8
20	9397	9387	9377	9367	9356	9346	1	2	3	4	5	6	7	8	9
21	9336	9325	9315	9304	9293	9283	1	2	3	4	5	6	7	8	9
22	9272	9261	9250	9239	9228	9216	1	2	3	4	6	6	7	9	10
23	9205	9194	9182	9171	9159	9147	1	2	3	5	6	7	8	9	10
24	9135	9124	9112	9100	9088	9075	1	2	4	5	6	7	8	9	10
25	9063	9051	9038	9026	9013	9001	1	3	4	5	6	7	8	10	11
26	8988	8975	8962	8949	8936	8923	1	3	4	5	6	8	9	10	11
27	8910	8897	8884	8870	8857	8843	1	3	4	5	7	8	9	10	12
28	8829	8816	8802	8788	8774	8760	1	3	4	6	7	8	9	11	12
29	8746	8732	8718	8704	8689	8675	1	3	4	6	7	8	10	11	12
30	8660	8646	8631	8616	8601	8587	1	3	4	6	7	9	10	11	13
31	8572	8557	8542	8526	8511	8496	2	3	5	6	8	9	10	12	13
32	8480	8465	8450	8434	8418	8403	2	3	5	6	8	9	11	12	14
33	8387	8371	8355	8339	8323	8307	2	3	5	6	8	9	11	12	14
34	8290	8274	8258	8241	8225	8208	2	3	5	7	8	10	11	13	14
35	8192	8175	8158	8141	8124	8107	2	3	5	7	8	10	12	13	15
36	8090	8073	8056	8039	8021	8004	2	3	5	7	9	10	12	14	15
37	7986	7969	7951	7934	7916	7898	2	4	5	7	9	10	12	14	16
38	7880	7862	7844	7826	7808	7790	2	4	5	7	9	11	12	14	16
39	7771	7753	7735	7716	7698	7679	2	4	6	7	9	11	13	14	16
40	7660	7642	7623	7604	7585	7566	2	4	6	8	9	11	13	15	17
41	7547	7528	7509	7490	7470	7451	2	4	6	8	10	11	13	15	17
42	7431	7412	7392	7373	7353	7333	2	4	6	8	10	12	13	15	17
43	7314	7294	7274	7254	7234	7214	2	4	6	8	10	12	14	16	18
44	7193	7173	7153	7132	7112	7092	2	4	6	8	10	12	14	16	18

Table V.

NATURAL COSINES

Deg	0′	10′	20′	30′	40′	50′	1	2	3	4	5	6	7	8	9
45	7071	7050	7030	7009	6988	6967	2	4	6	8	10	12	15	17	19
46	6947	6926	6905	6884	6862	6841	2	4	6	8	11	13	15	17	19
47	6820	6799	6777	6756	6754	6713	2	4	6	9	11	13	15	17	19
48	6691	6670	6648	6626	6604	6583	2	4	7	9	11	13	15	17	19
49	6561	6539	6517	6494	6472	6450	2	4	7	9	11	13	15	17	20
50	6428	6406	6383	6361	6338	6316	2	4	7	9	11	13	15	18	20
51	6293	6271	6248	6225	6202	6180	2	5	7	9	11	13	16	18	20
52	6157	6134	6111	6088	6065	6041	2	5	7	9	12	14	16	18	20
53	6018	5995	5972	5948	5925	5901	2	5	7	9	12	14	16	18	21
54	5878	5854	5831	5807	5783	5760	2	5	7	9	12	14	16	19	21
55	5736	5712	5688	5664	5640	5616	2	5	7	10	12	14	17	19	21
56	5592	5568	5544	5519	5495	5471	2	5	7	10	12	14	17	19	22
57	5446	5422	5398	5373	5348	5324	2	5	7	10	12	15	17	19	22
58	5299	5275	5250	5225	5200	5175	2	5	7	10	12	15	17	20	22
59	5150	5125	5100	5075	5050	5025	3	5	8	10	13	15	17	20	22
60	5000	4975	4950	4924	4899	4874	3	5	8	10	13	15	18	20	23
61	4848	4823	4797	4772	4746	4720	3	5	8	10	13	15	18	20	23
62	4695	4669	4643	4617	4592	4566	3	5	8	10	13	15	18	20	23
63	4540	4514	4488	4462	4436	4410	3	5	8	10	13	15	18	21	23
64	4384	4358	4331	4305	4279	4253	3	5	8	11	13	16	18	21	23
65	4226	4200	4173	4147	4120	4094	3	5	8	11	13	16	18	21	24
66	4067	4041	4014	3987	3961	3934	3	5	8	11	14	16	19	21	24
67	3907	3881	3854	3827	3800	3773	3	5	8	11	14	16	19	21	24
68	3746	3719	3692	3665	3638	3611	3	5	8	11	14	16	19	21	24
69	3584	3557	3529	3502	3475	3448	3	5	8	11	14	16	19	22	24
70	3420	3393	3365	3338	3311	3283	3	5	8	11	14	16	19	22	25
71	3256	3228	3201	3173	3145	3118	3	6	8	11	14	16	19	22	25
72	3090	3062	3035	3007	2979	2952	3	6	8	11	14	17	19	22	25
73	2924	2896	2868	2840	2812	2784	3	6	8	11	14	17	19	22	25
74	2756	2728	2700	2672	2644	2616	3	6	8	11	14	17	20	22	25
75	2588	2560	2532	2504	2476	2447	3	6	8	11	14	17	20	22	25
76	2419	2391	2363	2334	2306	2278	3	6	8	11	14	17	20	23	25
77	2250	2221	2193	2164	2136	2108	3	6	9	11	14	17	20	23	25
78	2079	2051	2022	1994	1965	1937	3	6	9	11	14	17	20	23	26
79	1908	1880	1851	1822	1794	1765	3	6	9	12	14	17	20	23	26
80	1736	1708	1679	1650	1622	1593	3	6	9	12	14	17	20	23	26
81	1564	1536	1507	1478	1449	1421	3	6	9	12	14	17	20	23	26
82	1392	1363	1334	1305	1276	1248	3	6	9	12	14	17	20	23	26
83	1219	1190	1161	1132	1103	1074	3	6	9	12	14	17	20	23	26
84	1045	1016	0987	0958	0929	0901	3	6	9	12	14	17	20	23	26
85	0872	0843	0814	0785	0756	0727	3	6	9	12	15	17	20	23	26
86	0698	0669	0640	0610	0581	0552	3	6	9	12	15	17	20	23	26
87	0523	0494	0465	0436	0407	0378	3	6	9	12	15	17	20	23	26
88	0349	0320	0291	0262	0233	0204	3	6	9	12	15	17	20	23	26
89	0175	0145	0116	0087	0058	0029	3	6	9	12	15	17	20	23	26

Table VI.

NATURAL TANGENTS

	0′	5′	10′	15′	20′	25′	30′	35′	40′	45′	50′	55′	1	2	3	4
0°	0000	0015	0029	0044	0058	0073	0087	0102	0116	0131	0145	0160	3	6	9	12
1	·0175	0189	0204	0218	0233	0247	0262	0276	0291	0306	0320	0335	3	6	9	12
2	0349	0364	0378	0393	0407	0422	0437	0451	0466	0480	0495	0509	3	6	9	12
3	·0524	0539	0553	0568	0582	0597	0612	0626	0641	0655	0670	0685	3	6	9	12
4	·0699	0714	0729	0743	0758	0772	0787	0802	0816	0831	0846	0860	3	6	9	12
5	·0875	0890	0904	0919	0934	0948	0963	0978	0992	1007	1022	1036	3	6	9	12
6	1051	1066	1080	1095	1110	1125	1139	1154	1169	1184	1198	1213	3	6	9	12
7	1228	1243	1257	1272	1287	1302	1317	1331	1346	1361	1376	1391	3	6	9	12
8	1405	1420	1435	1450	1465	1480	1495	1509	1524	1539	1554	1569	3	6	9	12
9	1584	1599	1614	1629	1644	1658	1673	1688	1703	1718	1733	1748	3	6	9	12
10	1763	1778	1793	1808	1823	1838	1853	1868	1883	1899	1914	1929	3	6	9	12
11	·1944	1959	1974	1989	2004	2019	2035	2050	2065	2080	2095	2110	3	6	9	12
12	2126	2141	2156	2171	2186	2202	2217	2232	2247	2263	2278	2293	3	6	9	12
13	·2309	2324	2339	2355	2370	2385	2401	2416	2432	2447	2462	2478	3	6	9	12
14	·2493	2509	2524	2540	2555	2571	2586	2602	2617	2633	2648	2664	3	6	9	12
15	2679	2695	2711	2726	2742	2758	2773	2789	2805	2820	2836	2852	3	6	9	13
16	2867	2883	2899	2915	2931	2946	2962	2978	2994	3010	3026	3041	3	6	9	13
17	3057	3073	3089	3105	3121	3137	3153	3169	3185	3201	3217	3233	3	6	10	13
18	3249	3265	3281	3298	3314	3330	3346	3362	3378	3395	3411	3427	3	6	10	13
19	3443	3460	3476	3492	3508	3525	3541	3558	3574	3590	3607	3623	3	6	10	13
20	·3640	3656	3673	3689	3706	3722	3739	3755	3772	3789	3805	3822	3	7	10	13
21	3839	3855	3872	3889	3906	3922	3939	3956	3973	3990	4006	4023	3	7	10	13
22	4040	4057	4074	4091	4108	4125	4142	4159	4176	4193	4210	4228	3	7	10	14
23	4245	4262	4279	4296	4314	4331	4348	4365	4383	4400	4417	4435	3	7	10	14
24	4452	4470	4487	4505	4522	4540	4557	4575	4592	4610	4628	4645	4	7	10	14
25	4663	4681	4699	4716	4734	4752	4770	4788	4806	4823	4841	4859	4	7	11	14
26	4877	4895	4913	4931	4950	4968	4986	5004	5022	5040	5059	5077	4	7	11	15
27	5095	5114	5132	5150	5169	5187	5206	5224	5243	5261	5280	5298	4	7	11	15
28	5317	5336	5354	5373	5392	5411	5430	5448	5467	5486	5505	5524	4	8	11	15
29	5543	5562	5581	5600	5619	5639	5658	5677	5696	5715	5735	5754	4	8	12	15
30	5774	5793	5812	5832	5851	5871	5890	5910	5930	5949	5969	5989	4	8	12	16
31	6009	6028	6048	6068	6088	6108	6128	6148	6168	6188	6208	6228	4	8	12	16
32	6249	6269	6289	6310	6330	6350	6371	6391	6412	6432	6453	6473	4	8	12	16
33	6494	6515	6536	6556	6577	6598	6619	6640	6661	6682	6703	6724	4	8	13	17
34	·6745	6766	6787	6809	6830	6851	6873	6894	6916	6937	6959	6980	4	9	13	17
35	·7002	7024	7046	7067	7089	7111	7133	7155	7177	7199	7221	7243	4	9	13	18
36	7265	7288	7310	7332	7355	7377	7400	7422	7445	7467	7490	7513	5	9	14	18
37	7536	7558	7581	7604	7627	7650	7673	7696	7720	7743	7766	7789	5	9	14	18
38	·7813	7836	7860	7883	7907	7931	7954	7978	8002	8026	8050	8074	5	10	14	19
39	8098	8122	8146	8170	8195	8219	8243	8268	8292	8317	8342	8366	5	10	15	20
40	8391	8416	8441	8466	8491	8516	8541	8566	8591	8617	8642	8667	5	10	15	20
41	·8693	8718	8744	8770	8796	8821	8847	8873	8899	8925	8952	8978	5	10	16	21
42	·9004	9030	9057	9083	9110	9137	9163	9190	9217	9244	9271	9298	5	11	16	21
43	9325	9352	9380	9407	9435	9462	9490	9517	9545	9573	9601	9629	6	11	17	22
44	9657	9685	9713	9742	9770	9798	9827	9856	9884	9913	9942	9971	6	11	17	23

Table VI.

NATURAL TANGENTS

	0′	5′	10′	15′	20′	25′	30′	35′	40′	45′	50′	55′	1 2	3 4
45°	1 000	0029	0058	0088	0117	0147	0176	0206	0235	0265	0295	0325	6 12	18 24
46	1 035	0365	0416	0446	0477	0507	0538	0569	0599	0630	0661	0692	6 12	18 25
47	1 072	0755	0786	0818	0850	0881	0913	0945	0977	1009	1041	1074	6 13	19 25
48	1 111	1139	1171	1204	1237	1270	1303	1336	1369	1403	1436	1470	7 13	20 26
49	1 150	1538	1571	1606	1640	1674	1708	1743	1778	1812	1847	1882	7 14	21 28
50	1 192	1953	1988	2024	2059	2095	2131	2167	2203	2239	2276	2312	7 14	22 29
51	1 235	2386	2423	2460	2497	2534	2572	2609	2647	2685	2723	2761	8 15	23 30
52	1 280	2838	2876	2915	2954	2993	3032	3072	3111	3151	3190	3230	8 16	23 31
53	1 327	3311	3351	3392	3432	3473	3514	3555	3597	3638	3680	3722	8 16	25 33
54	1 376	3806	3848	3891	3934	3976	4019	4063	4106	4150	4193	4237	9 17	26 34
55	1 428	4326	4370	4415	4460	4505	4550	4596	4641	4687	4733	4779	9 18	27 36
56	1 483	4872	4919	4966	5013	5061	5108	5156	5204	5253	5301	5350	10 19	29 38
57	1 540	5448	5497	5547	5597	5647	5697	5747	5798	5849	5900	5952	10 20	30 40
58	1 600	6055	6107	6160	6212	6265	6319	6372	6426	6479	6534	6588	11 21	32 43
59	1 664	6698	6753	6808	6864	6920	6977	7033	7090	7147	7205	7262	11 23	34 45
60	1 732	7379	7437	7496	7556	7615	7675	7735	7796	7856	7917	7979	12 24	36 48
61	1 804	8103	8165	8228	8291	8354	8418	8482	8546	8611	8676	8741	13 26	38 51
62	1 881	8873	8940	9007	9074	9142	9210	9278	9347	9416	9486	9556	14 27	41 55
63	1 963	9697	9768	9840	9912	9984	$\overline{0057}$	$\overline{0130}$	0204	0278	$\overline{0353}$	$\overline{0428}$	15 29	44 58
64	2 050	0579	0655	0732	0809	0887	0965	1044	1123	1203	1283	1364	16 31	47 63
65	2 144	1527	1609	1692	1775	1859	1943	2028	2113	2199	2286	2373	17 34	51 68
66	2 246	2549	2637	2727	2817	2907	2998	3090	3183	3276	3369	3464	18 37	55 74
67	2 356	3654	3750	3847	3945	4043	4142	4242	4342	4443	4545	4648	20 40	60 79
68	2 475	4855	4960	5065	5172	5279	5386	5495	5605	5715	5826	5938	22 43	65 87
69	2 605	6165	6279	6395	6511	6628	6746	6865	6985	7106	7228	7351	24 47	71 95
70	2 747	2 760	2 773	2 785	2 798	2 811	2 824	2 837	2 850	2 864	2 877	2 891	3 5	8 10
71	2 904	2 918	2 932	2 946	2 960	2 974	2 989	3 003	3 018	3 033	3 047	3 063	3 6	9 11
72	3 078	3 093	3 108	3 124	3 140	3 156	3·172	3 188	3 204	3·221	3 237	3 254	3 6	10 13
73	3 271	3 288	3 305	3 323	3 340	3 358	3 376	3 394	3 412	3 431	3 450	3 468	4 7	11 14
74	3 487	3 507	3 526	3 546	3 566	3 586	3 606	3 626	3 647	3 668	3·689	3 710	4 8	12 16
75	3 732	3 754	3 776	3 798	3 821	3 844	3 867	3 890	3 914	3 938	3 962	3·986	5 9	14 19
76	4 011	4 036	4 061	4·087	4 113	4 139	4 165	4 192	4 219	4 247	4 275	4 303	5 11	16 21
77	4 331	4 360	4 390	4 419	4 449	4 480	4 511	4 542	4 574	4 606	4 638	4 671	6 12	19 25
78	4 705	4 739	4 773	4 808	4 843	4 879	4 915	4 952	4 989	5 027	5 066	5 105	7 15	22 29
79	5 145	5 185	5 226	5 267	5 309	5 352	5 396	5 440	5 485	5 530	5 576	5 623	9 17	26 35
80	5 671	5 720	5 769	5 820	5 871	5 923	5 976	6 030	6 084	6 140	6 197	6 255		
81	6·314	6 374	6 435	6 497	6 561	6 625	6 691	6 758	6 827	6 897	6 968	7 041		
82	7 115	7 191	7 269	7 348	7 429	7 511	7 596	7 682	7 770	7 861	7 953	8 048		
83	8 144	8 243	8 345	8 449	8 556	8 665	8 777	8 892	9·010	9 131	9 255	9 383		
84	9 514	9 649	9 788	9 931	10 08	10 23	10 39	10 55	10 71	10 88	11·06	11 24	Difference columns cease to be useful	
85	11 43	11 62	11 83	12·03	12 25	12 47	12 71	12 95	13 20	13 46	13 73	14 01		
86	14 30	14·61	14 92	15 26	15·60	15·97	16 35	16 75	17 17	17 61	18·07	18 56		
87	19 08	19 63	20 21	20 82	21 47	22 16	22 90	23 69	24 54	25 45	26 43	27 49		
88	28·64	29 88	31 24	32 73	34 37	36 18	38 19	40 44	42 96	45 83	49 10	52 88		
89	57 29	62 50	68 75	76 39	85 94	98 22	114 6	137 5	171·9	229 2	343 8	687·5		

Table VII.

RADIAN MEASURE OF ANGLES.

Deg	0′	10′	20′	30′	40′	50′
0	0·0000	0029	0058	0087	0116	0145
1	0·0175	0204	0233	0262	0291	0320
2	0·0349	0378	0407	0436	0465	0495
3	0·0524	0553	0582	0611	0640	0669
4	0·0698	0727	0756	0785	0814	0844
5	0·0873	0902	0931	0960	0989	1018
6	0·1047	1076	1105	1134	1164	1193
7	0·1222	1251	1280	1309	1338	1367
8	0·1396	1425	1454	1484	1513	1542
9	0·1571	1600	1629	1658	1687	1716
10	0·1745	1774	1804	1833	1862	1891
11	0·1920	1949	1978	2007	2036	2065
12	0·2094	2123	2153	2182	2211	2240
13	0·2269	2298	2327	2356	2385	2414
14	0·2443	2473	2502	2531	2560	2589
15	0·2618	2647	2676	2705	2734	2763
16	0·2793	2822	2851	2880	2909	2938
17	0·2967	2996	3025	3054	3083	3113
18	0·3142	3171	3200	3229	3258	3287
19	0·3316	3345	3374	3403	3432	3462
20	0·3491	3520	3549	3578	3607	3636
21	0·3665	3694	3723	3752	3782	3811
22	0·3840	3869	3898	3927	3956	3985
23	0·4014	4043	4072	4102	4131	4160
24	0·4189	4218	4247	4276	4305	4334
25	0·4363	4392	4422	4451	4480	4509
26	0·4538	4567	4596	4625	4654	4683
27	0·4712	4741	4771	4800	4829	4858
28	0·4887	4916	4945	4974	5003	5032
29	0·5061	5091	5120	5149	5178	5207
30	0·5236	5265	5294	5323	5352	5381
31	0·5411	5440	5469	5498	5527	5556
32	0·5585	5614	5643	5672	5701	5730
33	0·5760	5789	5818	5847	5876	5905
34	0·5934	5963	5992	6021	6050	6080
35	0·6109	6138	6167	6196	6225	6254
36	0·6283	6312	6341	6370	6400	6429
37	0·6458	6487	6516	6545	6574	6603
38	0·6632	6661	6690	6720	6749	6778
39	0·6807	6836	6865	6894	6923	6952
40	0·6981	7010	7039	7069	7098	7127
41	0·7156	7185	7214	7243	7272	7301
42	0·7330	7359	7389	7418	7447	7476
43	0·7505	7534	7563	7592	7621	7650
44	0·7679	7709	7738	7767	7796	7825

Difference for	18
1′	3
2′	6
3′	9
4′	12
5′	15
6′	18
7′	21
8′	24
9′	27

Table VII.

RADIAN MEASURE OF ANGLES

Deg	0′	10′	20′	30	40′	50′
45	0 7854	7883	7912	7941	7970	7999
46	0 8029	8058	8087	8116	8145	8174
47	0 8203	8232	8261	8290	8319	8348
48	0 8378	8407	8436	8465	8494	8523
49	0 8552	8581	8610	8639	8668	8698
50	0 8727	8756	8785	8814	8843	8872
51	0 8901	8930	8959	8988	9018	9047
52	0 9076	9105	9134	9163	9192	9221
53	0 9250	9279	9308	9338	9367	9396
54	0 9425	9454	9483	9512	9541	9570
55	0 9599	9628	9657	9687	9716	9745
56	0 9774	9803	9832	9861	9890	9919
57	0 9948	9977	0007	0036	0065	0094
58	1 0123	0152	0181	0210	0239	0268
59	1 0297	0327	0356	0385	0414	0443
60	1·0472	0501	0530	0559	0588	0617
61	1 0647	0676	0705	0734	0763	0792
62	1 0821	0850	0879	0908	0937	0966
63	1 0996	1025	1054	1083	1112	1141
64	1 1170	1199	1228	1257	1286	1316
65	1 1345	1374	1403	1432	1461	1490
66	1 1519	1548	1577	1606	1636	1665
67	1 1694	1723	1752	1781	1810	1839
68	1 1868	1897	1926	1956	1985	2014
69	1 2043	2072	2101	2130	2159	2188
70	1 2217	2246	2275	2305	2334	2363
71	1 2392	2421	2450	2479	2508	2537
72	1 2566	2595	2625	2654	2683	2712
73	1 2741	2770	2799	2828	2857	2886
74	1 2915	2945	2974	3003	3032	3061
75	1 3090	3119	3148	3177	3206	3235
76	1·3265	3294	3323	3352	3381	3410
77	1 3439	3468	3497	3526	3555	3584
78	1 3614	3643	3672	3701	3730	3759
79	1 3788	3817	3846	3875	3904	3934
80	1 3963	3992	4021	4050	4079	4108
81	1 4137	4166	4195	4224	4254	4283
82	1 4312	4341	4370	4399	4428	4457
83	1 4486	4515	4544	4573	4603	4632
84	1 4661	4690	4719	4748	4777	4806
85	1 4835	4864	4893	4923	4952	4981
86	1 5010	5039	5068	5097	5126	5155
87	1 5184	5213	5243	5272	5301	5330
88	1 5359	5388	5417	5446	5475	5504
89	1 5533	5563	5592	5621	5650	5679

Difference	
for	18
1′	3
2′	6
3′	9
4′	12
5′	15
6′	18
7′	21
8′	24
9′	27

Table VIII.

CHORDS OF ANGLES.

Deg	0′	10′	20′	30′	40′	50′	Deg	0′	10′	20′	30′	40′	50′
0	·000	003	006	009	012	014	**45**	765	768	771	773	·776	779
1	017	020	023	026	029	032	46	781	784	787	789	792	795
2	035	038	041	·044	046	049	47	797	800	803	805	808	811
3	052	055	058	061	064	067	48	813	816	819	821	824	827
4	070	073	·076	·078	081	084	49	829	832	835	837	840	843
5	·087	090	·093	096	·099	102	**50**	·845	848	850	853	856	858
6	·105	108	110	113	116	119	51	861	864	866	869	871	874
7	122	125	128	131	134	137	52	·877	879	882	885	887	890
8	·139	·142	145	148	151	154	53	892	895	898	900	903	905
9	157	160	163	166	168	171	54	908	911	913	916	918	921
10	174	177	180	183	186	189	**55**	923	926	929	931	·934	·936
11	192	195	197	200	203	206	56	939	941	944	·947	949	952
12	209	212	215	218	221	223	57	954	957	959	962	964	967
13	226	229	232	·235	238	241	58	970	972	·975	977	980	982
14	244	247	249	252	255	258	59	985	987	990	992	995	997
15	261	264	267	270	273	275	**60**	1 000	1 002	1 005	1 007	1 010	1 013
16	278	281	284	287	290	293	61	1 015	1 018	1 020	1 023	1 025	1 028
17	296	·298	301	304	307	310	62	1 030	1 033	1 035	1 037	1 040	1 042
18	313	·316	319	321	324	327	63	1 045	1 047	1 050	1 052	1 055	1 057
19	·330	·333	336	339	342	344	64	1 060	1 062	1 065	1 067	1 070	1 072
20	·347	350	353	356	359	362	**65**	1 075	1 077	1 079	1 082	1 084	1 087
21	364	367	370	373	376	379	66	1 089	1 092	1 094	1 097	1 099	1 101
22	382	384	387	390	393	396	67	1 104	1 106	1 109	1 111	1 113	1 116
23	399	402	404	407	410	413	68	1 118	1 121	1 123	1 126	1 128	1 130
24	416	419	421	424	427	430	69	1 133	1 135	1 138	1 140	1 142	1 145
25	433	436	438	441	444	447	**70**	1 147	1 149	1 152	1 154	1 157	1 159
26	450	453	456	458	461	464	71	1 161	1 164	1 166	1 168	1 171	1 173
27	467	470	472	475	478	481	72	1 176	1 178	1 180	1 183	1 185	1 187
28	484	487	489	492	495	498	73	1 190	1 192	1 194	1 197	1 199	1 201
29	501	504	506	509	512	515	74	1 204	1 206	1 208	1 211	1 213	1 215
30	518	520	523	526	529	532	**75**	1 217	1 220	1 222	1 224	1 227	1 229
31	534	537	540	543	546	548	76	1 231	1 234	1 236	1 238	1 240	1 243
32	551	554	·557	560	562	565	77	1 245	1 247	1 250	1 252	1 254	1 256
33	·568	571	574	576	579	582	78	1 259	1 261	1 263	1 265	1 268	1 270
34	585	587	590	593	596	599	79	1 272	1 274	1 277	1 279	1 281	1 283
35	·601	604	607	610	612	·615	**80**	1 286	1 288	1 290	1 292	1 294	1 297
36	618	621	·624	626	629	632	81	1 299	1 301	1 303	1 305	1 308	1 310
37	·635	637	640	643	646	648	82	1 312	1 314	1 316	1 319	1 321	1 323
38	651	·654	657	659	662	665	83	1 325	1 327	1 330	1 332	1 334	1 336
39	·668	670	673	676	679	681	84	1 338	1 340	1 343	1 345	1 347	1 349
40	684	687	·689	692	695	698	**85**	1 351	1 353	1 355	1 358	1 360	1 362
41	700	703	706	709	711	714	86	1 364	1 366	1 368	1 370	1 372	1·375
42	·717	·719	·722	725	728	730	87	1 377	1 379	1 381	1 383	1 385	1 387
43	733	736	·738	741	744	746	88	1 389	1 391	1 393	1·396	1·398	1 400
44	·749	752	755	·757	760	763	89	1 402	1 404	1 406	1 408	1 410	1·412
							90	1 414					

Table IX.

Angle Deg	Angle Radians	Chords	Sine.	Tangent	Cotangent	Cosine			
0°	0	0	0	0	∞	1	1.414	1.5708	**90°**
1	.0175	.017	.0175	.0175	57.2900	.9998	1.402	1.5533	89
2	.0349	.035	.0349	.0349	28.6363	.9994	1.389	1.5359	88
3	.0524	.052	.0523	.0524	19.0811	.9986	1.377	1.5184	87
4	.0698	.070	.0698	.0699	14.3006	.9976	1.364	1.5010	86
5	.0873	.087	.0872	.0875	11.4301	.9962	1.351	1.4835	**85**
6	.1047	.105	.1045	.1051	9.5144	.9945	1.338	1.4661	84
7	.1222	.122	.1219	.1228	8.1443	.9925	1.325	1.4486	83
8	.1396	.139	.1392	.1405	7.1154	.9903	1.312	1.4312	82
9	.1571	.157	.1564	.1584	6.3138	.9877	1.299	1.4137	81
10	.1745	.174	.1736	.1763	5.6713	.9848	1.286	1.3963	**80**
11	.1920	.192	.1908	.1944	5.1446	.9816	1.272	1.3788	79
12	.2094	.209	.2079	.2126	4.7046	.9781	1.259	1.3614	78
13	.2269	.226	.2250	.2309	4.3315	.9744	1.245	1.3439	77
14	.2443	.244	.2419	.2493	4.0108	.9703	1.231	1.3265	76
15	.2618	.261	.2588	.2679	3.7321	.9659	1.217	1.3090	**75**
16	.2793	.278	.2756	.2867	3.4874	.9613	1.204	1.2915	74
17	.2967	.296	.2924	.3057	3.2709	.9563	1.190	1.2741	73
18	.3142	.313	.3090	.3249	3.0777	.9511	1.176	1.2566	72
19	.3316	.330	.3256	.3443	2.9042	.9455	1.161	1.2392	71
20	.3491	.347	.3420	.3640	2.7475	.9397	1.147	1.2217	**70**
21	.3665	.364	.3584	.3839	2.6051	.9336	1.133	1.2043	69
22	.3840	.382	.3746	.4040	2.4751	.9272	1.118	1.1868	68
23	.4014	.399	.3907	.4245	2.3559	.9205	1.104	1.1694	67
24	.4189	.416	.4067	.4452	2.2460	.9135	1.089	1.1519	66
25	.4363	.433	.4226	.4663	2.1445	.9063	1.075	1.1345	**65**
26	.4538	.450	.4384	.4877	2.0503	.8988	1.060	1.1170	64
27	.4712	.467	.4540	.5095	1.9626	.8910	1.045	1.0996	63
28	.4887	.484	.4695	.5317	1.8807	.8829	1.030	1.0821	62
29	.5061	.501	.4848	.5543	1.8040	.8746	1.015	1.0647	61
30	.5236	.518	.5000	.5774	1.7321	.8660	1.000	1.0472	**60**
31	.5411	.534	.5150	.6009	1.6643	.8572	.985	1.0297	59
32	.5585	.551	.5299	.6249	1.6003	.8480	.970	1.0123	58
33	.5760	.568	.5446	.6494	1.5399	.8387	.954	.9948	57
34	.5934	.585	.5592	.6745	1.4826	.8290	.939	.9774	56
35	.6109	.601	.5736	.7002	1.4281	.8192	.923	.9599	**55**
36	.6283	.618	.5878	.7265	1.3764	.8090	.908	.9425	54
37	.6458	.635	.6018	.7536	1.3270	.7986	.892	.9250	53
38	.6632	.651	.6157	.7813	1.2799	.7880	.877	.9076	52
39	.6807	.668	.6293	.8098	1.2349	.7771	.861	.8901	51
40	.6981	.684	.6428	.8391	1.1918	.7660	.845	.8727	**50**
41	.7156	.700	.6561	.8693	1.1504	.7547	.829	.8552	49
42	.7330	.717	.6691	.9004	1.1106	.7431	.813	.8378	48
43	.7505	.733	.6820	.9325	1.0724	.7314	.797	.8203	47
44	.7679	.749	.6947	.9657	1.0355	.7193	.781	.8029	46
45	.7854	.765	.7071	1.0000	1.0000	.7071	.765	.7854	**45**
			Cosine	Cotangent	Tangent	Sine	Chords	Angle Radians.	Angle Deg

EXAMINATION PAPERS OF THE BOARD OF EDUCATION.

ADVANCED PRACTICAL MATHEMATICS 1901.

Only EIGHT *questions to be answered.*

1. Compute

$30\ 56 \div 4{\cdot}105,\quad 0\ 03056 \times 0{\cdot}4105,\quad 4{\cdot}105^{1{\cdot}23},\quad 0\ 04105^{-2\ 3}.$

The answers must be right to three significant figures

Why do we multiply $\log a$ by b to obtain the logarithm of a^b?

2 If $a=5$, $b=200$, $c=600$, $g=0\ 1745$ radian, find the value of

$$ae^{-bt}\sin(ct+g)$$

(i) When $t=0\ 001$, (ii) when $t=0\ 01$, (iii) when $t=0{\cdot}1$.

3. The keeper of a restaurant finds when he has G guests in a day, his total daily expenditure is E pounds (for rent, taxes, wages, wear and tear, food and drink), and his total daily receipt is R pounds The following numbers are averages obtained by examination of his books on many days ·

G	E	R
210	16·7	15 8
270	19 4	21·2
320	21 6	26 4
360	23 4	29 8

Using squared paper, find E and R and the day's profits if he has 340 guests.

What number of guests per day just gives him no profit?

What simple algebraical laws seem to connect E, R, P, the profit, and G?

Two of the marks will be given for a correct answer to the following:

If he finds that he has almost too many guests from, say, 1 to 2 o'clock, and from, say, 6 to 7 o'clock, and almost none at other times of the day, what expedient might he adopt to increase his profits?

4. The following quantities are thought to follow a law like pv^n = constant Try if they do so, find the most probable value of n

v	1	2	3	4	5
p	205	114	80	63	52

5. There is a curve whose shape may be drawn from the following values of x and y

x in feet	3	3 5	4 2	4 8
y in inches	10 1	12 2	13 1	11 9

Imagine this curve to rotate about the axis of x describing a surface of revolution. What is the volume enclosed by this surface and the two end sections where $x=3$ and $x=4\ 8$?

6 If $x=a \sin pt+b \cos pt$ for any value of t where a, b, and p are mere numbers, show that this is the same as $x=A \sin (pt+e)$ if A and e are properly evaluated

7 Let a closed curve rotate round a straight line in its own plane and generate a ring; state and prove the two rules for finding the volume and surface of the ring

8 Two sides of a triangle are measured and found to be 32·5 and 24 2 inches; the included angle being 57°, find the area of the triangle. Prove the rule used by you. If the true lengths of the sides are really 32 6 and 24 1, what is the percentage error in the answer?

9. The polar co-ordinates of a point are $r=5$ feet, $\theta=52°$; $\phi=70°$, find the x, y, and z co-ordinates, also find the angles made by r with the axes of co-ordinates

10. Define carefully what is meant by the Scalar Product of two vectors and by the Vector Product of two vectors, giving one useful example of each

11. There is a piece of mechanism whose weight is 200 lbs The following values of s in feet show the distance of its centre of gravity (as measured on a skeleton drawing) from some point in its straight path at the time t seconds from some era of reckoning Find its acceleration at the time $t=2\ 05$, and the force in pounds which is giving this acceleration to it

s	0·3090	0 4931	0 6799	0·8701	1 0643	1 2631
t	2 0	2 02	2 04	2 06	2 08	2 10

12. What is meant by the symbol $\frac{dy}{dx}$? Explain how it may be represented by the slope of a curve. State its value in the cases

$$y=ax^n,\quad y=a\epsilon^{bx},\quad y=a\sin(bx+c),$$
$$y=a\cos(bx+c),\quad y=\log_e(x+b).$$

13. Find $\int p\ dv$, if $pv^s=c$, a constant,

(1) when $s=0.8$,

(2) when $s=1$

14. In the curve $y=cx^{\frac{1}{2}}$, find c if $y=m$ when $x=b$ Let this curve rotate about the axis of x; find the volume enclosed by the surface of revolution between the two sections at $x=a$ and $x=b$ Of course, m, b, and a are given distances

15. The rate (per unit increase of volume) of reception of heat by a gas is h, p is its pressure, and v its volume, γ is a known constant If $pv^s=c$, s and c being constants, find h if

$$h=\frac{1}{\gamma-1}\left\{v\frac{dp}{dv}+\gamma p\right\}$$

Full marks will be given only when the answer is stated in its simplest form

If h is always 0, find what s must be.

16 At the following draughts in sea water a particular vessel has the following displacements

Draught h feet, - -	15	12	9	6 3
Displacement T tons, -	2098	1512	1018	586

Plot $\log T$ and $\log h$ on squared paper, and try to get a simple rule connecting T and h If one ton of sea water measures 35 cubic feet, find the rule connecting V and h, if V is the displacement in cubic feet

17 Preferably to be answered by a Candidate who has already answered Question 16 Find how A the horizontal sectional area of the vessel at the water line depends upon h At any draught h what change of displacement V or T is produced by one *inch* difference in h?

18. In any class of turbine if P is the power of the waterfall and H the height of the fall, n the rate of revolution, and R is the average radius at the place where water enters the wheel, then it is known that for any particular class of turbines of all sizes

$$n \propto H^{0.25}P^{-0.5},$$
$$R \propto P^{0.5}H^{-0.75}.$$

In the list of a particular maker I take a turbine at random for a fall of 6 feet, 100 horse-power, 50 revolutions per minute, 2 51 feet radius By means of this I find I can calculate n and R for all the other turbines of the list Find n and R for a fall of 20 feet and 75 horse-power.

ADVANCED PRACTICAL MATHEMATICS 1902.

Only EIGHT *questions are to be answered Three of these must be Nos.* 1, 2, and 3

1 Compute by contracted methods, without using logarithms,

$$23\ 07 \times 0\ 1354, \quad 2307 - 1\ 354$$

Compute $2\ 307^{0.65}$ and $23\ 07^{-1.25}$ using logarithms The answers to consist of four significant figures

Why do we add logarithms to obtain the logarithm of a product?

Suppose we have a scale on a slide rule on which, as usual, the distance to any mark n is $\log n$, and there is another scale on which the distance to any mark m is $\log(\log m)$, show that we can at once read off m^n and also the logarithm of any number to any base

2 Write in a table the values of the sine, cosine, and tangent of the following angles ·

$$23°, 123°, 233°, 312°, 383°.$$

3 What is meant by the symbol $\frac{dy}{dx}$?

Explain how it may be represented by the slope of a curve

If $y = 2\ 4 - 1\ 2x + 0\ 2x^2$ find $\frac{dy}{dx}$ and plot two curves from $x=0$ to $x=4$, showing how y and $\frac{dy}{dx}$ depend upon x.

4 Work the following three exercises as if in each case one were alone given, taking in each case the simplest supposition which your information permits

(*a*) The total yearly expense in keeping a school of 100 boys is £2,100, what is the expense when the number of boys is 175?

(*b*) The expense is £2,100 for 100 boys, £3,050 for 200 boys; what is it for 175 boys?

(*c*) The expenses for three cases are known as follows:

£2,100 for 100 boys,
£2,650 for 150 boys,
£3,050 for 200 boys

What is the probable expense for 175 boys?

If you use a squared paper method, show all three solutions together.

5 For the years 1896-1900, the following average numbers are taken from the accounts of the 34 most important Electric Companies of the United Kingdom.

U means millions of units of electric energy sold to customers C means the total cost in millions of pence, and includes interest (7 per cent.) on capital, maintenance, rent, taxes, salaries, wages, coal, etc.

U	0·67	1·00	1·366	1·46	2·49
C	4·84	6·25	8·60	9·11	14·25

Is there any simple approximately correct law connecting U and C? If so, what is it? Assume that from the beginning there was the idea of, at some time, reaching a maximum output of 13·9, so that $U \div 13{\cdot}9$ is called f, a certain kind of *load factor*. Let $C \div U$ be called c the total cost per unit; is there any law connecting c and f? You need not plot c and f; it is better to use the law already found.

6 In some experiments in towing a canal boat the following observations were made, P being the pull in pounds and v the speed of the boat in miles per hour

v	1·68	2·43	3·18	3·60	4·03
P	76	160	240	320	370

Plot $\log v$ and $\log P$ upon squared paper and give an approximate formula connecting P and v

7. What is the idea on which compound interest is calculated? Explain, as if to a beginner, how it is that

$$A = P\left(1 + \frac{r}{100}\right)^n,$$

where P is the money lent and A is what it amounts to in n years at r per cent per annum If A is 130 and P is 100 and n is 7·5, find r What does the above equation become when we imagine interest to be added on to principal every instant? State two natural phenomena which follow the compound interest law

8 Only *one* of the following, (a) or (b), is to be attempted.

(a) The inside diameter of a hollow sphere of cast iron is the fraction 0·57 of its outside diameter. Find these diameters if the weight is 60 lb Take one cubic inch of cast iron as weighing 0·26 lb.

If the outside diameter is made 1 per cent smaller, the inside not being altered, what is the percentage diminution of weight?

(*b*) The cross-section of a ring is an ellipse whose principal diameters are 2 inches and $1\frac{1}{2}$ inches, the middle of this section is at 3 inches from the axis of the ring, what is the volume of the ring?
Prove the rule you use for finding the volume of any ring

9 If pv^k is constant; and if $p=1$ when $v=1$, find for what value of v, p is 0 2. Do this for the following values of k, 0 8, 0 9, 1 0, 1 1 Tabulate your answers.

10 Define carefully what is meant by the Scalar Product and by the Vector Product of two vectors, giving one useful example of each

11 There is a point P whose x, y and z co-ordinates are 2, 1 5 and 3 Find its r, θ and ϕ co-ordinates If O is the origin, find the angles made by OP with the axes of co-ordinates

12 When is $x^{\frac{2}{\gamma}} - x^{1+\frac{1}{\gamma}}$ a maximum, γ being 1 4? Plot the values near the maximum value For this purpose you need calculate only the maximum value and two others

13 If the current C ampères in a circuit follows the law $C=10\sin 600t$; if t is in seconds, and if

$$V=RC+L\frac{dC}{dt},$$

where R is 0 3 and L is 4×10^{-4}, what is V?
Show by a sketch how C and V depend upon time, and particularly how one lags behind the other, and also state their highest and lowest values

14. There is a function

$$y=5\log_{10}x+6\sin\tfrac{1}{10}x+0\ 084(x-3\ 5)^2$$

Find a much simpler function of x which does not differ from it in value more than 2 per cent between $x=3$ and $x=6$ Remember that the angle $\frac{1}{10}x$ is in radians

ADVANCED PRACTICAL MATHEMATICS 1903

Only EIGHT *questions are to be answered. Three of these must be Nos* 1, 2, *and* 3.

1 Compute by contracted methods to four significant figures only, and without using logarithms or slide rule

$$8\ 102\times35\ 14,\quad 254\ 3-0\cdot09027$$

State the logarithms of 37240, 37 24, 0·03724
Compute, using logarithms,

$$\sqrt[3]{37\cdot24},\quad \sqrt[2]{3\cdot724},\quad 372\cdot4^{2\ 43},\quad 0\ 3724^{-2\ 43}$$

Explain why it is that logarithms are multiplied in computing the powers of numbers

In using your four-figure logarithm table have you observed that there is more chance of error at some places than at others? How is this? Can you suggest an improvement in such tables?

2 The three parts (*a*), (*b*), and (*c*) must be all answered to get full marks

(*a*) If $\theta=0\ 8\pi$, $\mu=0\ 3$ and $N=Me^{\mu\theta}$; if $(N-M)\ V=33000\ P$; if P is 30 and V is 520, find N

(*b*) Find the value of $10e^{-0\ 7t}\sin(2\pi ft+0\ 6)$, where f is 225 and t is 0 003

Observe that the angle is stated in *radians*

(*c*) If
$$A=P\left(1+\frac{r}{100}\right)^n,$$
and if $A=3P$ when $r=3\frac{1}{2}$, find n

3 $y=a+bx^n$ is the equation to a curve which passes through these three points,

$$x=0,\ y=1\ 24;\ x=2\ 2,\ y=5\ 07;\ x=3\ 5,\ y=12\ 64,$$

find a, b, and n

When we say that $\frac{dy}{dx}$ is shown by the slope of the curve, what exactly do we mean? Find $\frac{dy}{dx}$ when $x=2$

4 The following are the areas of cross section of a body at right angles to its straight axis.

A in square inches, - -	250	292	310	273	215	180	135	120
x inches from one end, -	0	22	41	70	84	102	130	145

What is the whole volume from $x=0$ to $x=145$?

At $x=50$, if a cross-sectional slice of small thickness δx has the volume δv, find $\frac{\delta v}{\delta x}$

5. Find accurately to three significant figures, a value of x to satisfy the equation

$$0\ 5x^{1\ 5}-12\log_{10}x+2\sin 2x=0\ 921.$$

Notice in *sin* $2x$ that the angle is in radians

6 The population of a country was $4\ 35\times10^6$ in 1820, $7\ 5\times10^6$ in 1860, $11{\cdot}26\times10^6$ in 1890 Test if the population follows the compound interest law of increase. What is the probable population in 1910?

7 The following table records the growth in stature of a girl A (born January, 1890) and a boy B (born May, 1894). Plot these records. Heights were measured at intervals of four months.

TABLE OF HEIGHTS IN INCHES.

Year	1900	1901			1902			1903.
Month	Sept	Jan	May	Sept	Jan	May	Sept	Jan
A	54 75	55 55	56 6	57 95	59 2	60 2	60 9	61 3
B	48 25	49 0	49 75	50 6	51 5	52 3	53 1	53 9

Find in inches per annum, the average rates of growth of A and B during the whole period of tabulation What will be the probable heights of A and B at the end of another four months? Plot the *rate* of growth of A at all times throughout the period. At about what age was A growing most rapidly and what was her quickest rate of growth?

8. The New Zealand Pension law for a person who has already lived from the age of 40 to 65 in the colony is

If the private income I is not more that £34 a year, the pension P is £18 a year If the private income is anything from 34 to 52, the pension is such that the total income is just made up to 52. If the private income is 52 or more there is no pension

Show on squared paper, for any income I the value of P, and also the value of the total income. If a person's private income is say £50, how much of it has he an inducement to give away before he applies for a pension? Show on the same paper the total income, if the pension were regulated according to the rule

$$P = 18 - \tfrac{9}{26} I$$

9 The following table gives corresponding values of two quantities x and y

y	10 16	12 26	14 70	20 80	24 54	28 83
x	37 36	31 34	26 43	19 08	16 33	14·04

Try whether x and y are connected by a law of the form $yx^n = c$, and if so, determine as nearly as you can the values of n and c

What is the value of x when $y = 17\ 53$?

10 Both parts (a) and (b) must be answered to get full marks

(a) Prove the rules used in finding the volume and area of a ring The mean radius of a ring is 2 feet The cross-section of the ring is an ellipse whose major and minor diameters are 0·8 and 0 5 feet; what is its volume?

(b) The length of a plane closed curve is divided into 24 elements, each of 1 inch long. The middles of successive elements are at the

distances x from a line in the plane, as follows (in inches):—10, 10·5, 10·91, 11·24, 11·49, 11·67, 12·57, 11·67, 11·49, 11·24, 10·91, 10·5, 10, 10·5, 10·91, 11·24, 11·49, 11·67, 12·57, 11·67, 11·49, 11·24, 10·91, 10·5.

If the curve rotates about the line as an axis describing a ring, find approximately the area of the ring.

11. Three planes of reference, mutually perpendicular, meet at O. The distances of a point P from the three planes are $x=1·2$, $y=2·7$, $z=0·9$. The distances of a point Q are $x=0·8$, $y=1·8$, $z=1·5$.

Find 1st, the distances OP and OQ,

2nd, the distance PQ,

3rd, the angle between OP and OQ.

12. Find the moment of inertia of a hollow right circular cylinder, internal radius R_1, external R_0, length l, about the axis of figure.

Prove the rule by which, when we know the moment of inertia of a body about an axis through its centre of mass we find its moment of inertia about any parallel axis.

What is the moment of inertia of our hollow cylinder about an axis lying in its interior surface?

13. If the current C ampères in a circuit follows the law

$$C = 10 \sin 600t$$

where t is in seconds. If $V = RC + L\dfrac{dC}{dt}$

where $R=0·3$, $L=4\times10^{-4}$, find V.

Show by a sketch how C and V vary with the time t, and particularly how one lags behind the other, and also state their highest and lowest value.

14. The entropy ϕ ranks of a quantity of stuff at the absolute temperature t degrees is known to vary in the following way:

t	443	403	373	343
ϕ	1·584	1·668	1·749	1·850

Plot ϕ horizontally and t vertically.

A rectangle, whose dimension horizontally represents 0·1 rank and whose vertical dimension represents 10 degrees, has an area which represents 0·1 × 10 or 1 unit of heat, what heat does each square inch of your diagram represent? The total heat received from beginning to end of the above set of changes is represented by the total area between the curve, the two end verticals and the zero line of temperature; state the amount of it.

You need not, of course, plot the whole of ϕ, you may subtract, say, 1·5 from each of the values. Also, if you want greater accuracy and can estimate areas of rectangles not actually drawn, you need not plot the whole value of t.

ANSWERS.

Exercises I., p. 10.

1. $4x(x^2+1)\{(x^2+1)^2-x^2\}$; 3·5174 **2** 0·236.

3 $\dfrac{a^{\frac{3}{2}}}{8}+\dfrac{ab^{\frac{1}{2}}}{12}-\dfrac{a^{\frac{1}{2}}b}{18}-\dfrac{b^{\frac{3}{2}}}{27}$, 3·149 **4** $\dfrac{14x}{1-9x^2}$, $2\frac{1}{3}$

5 $\dfrac{x+2}{x^2+1}$; 0·5590 **6.** 5·268 **7** 3·46

8. 0·2397 **11** 0·2236; 0·0557 **12** $\dfrac{2}{ab}$

13 1·0557 **14** $8a^3$ **15.** $\dfrac{6\sqrt{x}+3x}{1+\sqrt{x}-2x}$, 3·4020

17 $\dfrac{4}{x^2-1}$, 0·6188 **18.** $-\dfrac{c}{e}$ **19** 1 **20.** $(4x-3y)(3x-4y)$.

21 $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$

22 $(x^2+y^2+xy+1)(x^2+y^2-xy-1)$

24. $(4x-5)(5x+6)$ **25** $(2y+7)(x+3)$ **26** $(5x-7)(x-3a)$

27 $(x-1)^2(x^2+2x+3)$ **28** a. **29** $x+1$.

30 0·9659. **31** 9 **34** $\dfrac{1}{x-2}+\dfrac{1}{x-3}$

35 $\dfrac{4}{1-3x}-\dfrac{5}{1-2x}$ **36.** $\dfrac{1}{x-1}-\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}$ **37.** $\dfrac{2}{x+3}-\dfrac{1}{x-5}$

38 $\dfrac{2}{x+1}-\dfrac{1}{x-2}$. **39** $\dfrac{4}{x-3}-\dfrac{3}{x+7}$ **40** $\dfrac{3}{x-3}+\dfrac{2}{x-4}$

41. $\dfrac{3x}{x-2}+\dfrac{2}{x-4}$. **42** $\dfrac{2x}{x-1}+\dfrac{3}{x-2}-\dfrac{4}{x-3}$. **43** $\dfrac{1}{x-1}-\dfrac{4}{x+1}$.

Exercises II., p. 20.

1. $\dfrac{7\pi}{32}$. **2.** 47° 45′ **3** $\dfrac{4\pi}{3}$, 120°.

4. 1·0872, 0·9128 **5** 435·7 **6** 0·3927 miles.

7. 0·7431, −0·6947, −0·6745. **8.** 0·2588, −0·6691, 0·3249.

9.

angle	23°	123°	233°	312°	383°
sine	0·3907	0·8387	−0·7986	−0·7431	0·3907
cosine	0·9205	−0·5446	−0·6018	0·6691	0·9205
tangent	0·4245	1·5399	1·3270	−1·1106	0·4245

6·702 radians

11 43° 35′, 136° 25′

Exercises III., p. 35.

1 $\frac{63}{65}$; $\frac{16}{65}$ 2. $\frac{63}{65}$; $-\frac{16}{65}$. 6 0·6561

7 0·9898; $\frac{1}{2}$. 8 0·28, 0·96 10 $\frac{\sqrt{7}}{4}$, $\frac{\sqrt{7}}{3}$; $\frac{3\sqrt{7}}{7}$.

19 18·72 20 −0·39; 113 22 7 26 $-\frac{7}{25}$, $\frac{24}{25}$, $\frac{2}{5}\sqrt{5}$.

Exercises IV., p. 41.

1 60°, 120° 2 45°, 135° 3 30°, 150°.

4 45°, 71° 33′ 5 120°. 6 60°, 15°, etc.

7 (i) 52° 1′, 127° 58′, (ii) 134° 45′, (iii) 70° 52′, 160° 52′

8 45° 9 70° 32′ 10 120°, 0°

11 30°, 60° 12 30°, 150° 13 45°, 60°.

14 90°, 45° 15 216° 52′ 16 270°

17 69° 18′ 18 (i) 120°; (ii) 135°, (iii) 13° 20′.

19. 45° 60°, 120° 20 −0·4446, −0·4446.

21. 28° 9′, 61° 51′, 118° 9′, 151° 51′

22 71° 2′, 108° 58′, 251°, 288° 58′

23. $A = 39°$ 48′, $B = 27°$ 54′ 24 38° 30′.

25. 29° 17′ 26 45° 27 122° 18′.

28 54, 126°. 30 (a) 60°, (b) 30°

31 30° 32 19° 15′, 70° 45′, etc

34. 19° 9′. 36 8° 9′ 37. $\frac{\pi}{4}$, $\frac{\pi}{5}$, $\frac{2\pi}{3}$

Exercises V., p. 47.

1. $-\frac{3}{16}$ 3 $x^{\frac{3}{n}} - x^{-\frac{3}{n}}$ 4 $x^6 + \frac{1}{x^6} + 3\left(x^2 + \frac{1}{x^2}\right)$

5. $3 + 2x^{-\frac{1}{4}}y^{\frac{1}{4}} + x^{-\frac{1}{2}}y^{\frac{1}{2}} + 2x^{\frac{1}{4}}y^{-\frac{1}{4}} + x^{\frac{1}{2}}y^{-\frac{1}{2}}$ 6 $a^3bc^{\frac{1}{6}}$

7. $\frac{1}{5^3}$. 8. $x=2$, $y=3$. 9. (b) $x^2y^{\frac{7}{2}}$, (c) 1.

10 $x^{\frac14}y^{\frac34}(x^{\frac34}-4x^{\frac12}y^{\frac14}+16x^{\frac14}y^{\frac12}-16y^{\frac34})$		**11.** $a^{\frac32}+b+c^{\frac34}-3a^{\frac12}b^{\frac13}c^{\frac14}$.
12. 12	**13** 1·285	**14** $a^{-\frac{27}{16}}b^{-\frac{7}{12}}$
15 $a^{1\frac12}b^2$	**16.** (i) $\left(\frac{p}{q}\right)^{\frac23}+\left(\frac{p}{q}\right)^{\frac13}+\left(\frac{q}{p}\right)^{\frac13}+\left(\frac{q}{p}\right)^{\frac23}$, $x^{-1}y^{\frac16}$	
17 x^2	**18** $a^{\frac43}+a^{\frac23}b^{\frac32}+b^3$; 84·96	**19** 25·2

Exercises VI., p. 62.

1. 0·5535	**2** 3·123, 1704	**3** 12
4. 0·3722	**5** (i) 0·4722, (ii) 0·9563	**6** $\frac{9}{32}$.
7. 15·5	**8** 1·7022.	**9** 55
10 303.	**11** $6·506\times10^5$.	**12** 3·514
13 1·027.	**14** 1132.	**15** 245·5, 280.
16 $p=0·4286$, 0·3952, 0·3642; $v=3$, 2·306, 2·643		
17 14407, 16604, 18557, 18815		**18** 1·722, 0·0198.
19 $\sqrt{10}$.	**20** 39·98	**21** 254·6.
22. 75·06.	**23.** $x=0·9625$, $y=0·5668$	
24 0·5.	**25** 2·078 .	**26** 0·2184, 0·5986.
27 3·17.	**28** 2·885.	**29** $29·2\times10^6$.
30 1·613	**31** 0·9895.	**32.** 0·1556, 2100, 1·2810.
33 (i) 0·4315, (ii) $\bar{4}·8596$; (iii) $\bar{1}·8210$		**35** $-0·8899$
36 $G=4·516$, $D=3$	**37** 1·0275	**38** $-2·89$.
39 (i) $V=48·5$, $v=59·28$; (ii) $V=76·92$, $v=98$		**40.** 0·9266.
41. (i) 33280;	(ii) 33570;	(iii) 35850;
(iv) 33440;	(v) 44910;	(vi) 40500
42. 31518 gallons	**43.** $-9·897$	**44** 50·9443
45 133·6. **46** 3·042	**47** 1·8136, 4·265, ∞	**48.** 0·5491.

Exercises VII., p. 71.

1 30	**2** 11	**3.** $1\frac12$.	**4** 13
5 $3\frac12$.	**6** $\frac23$	**7.** 106.	**8** 111.
9 $\frac13$.	**10** -4	**11** 3.	**12** 1
13 7	**14** $4\frac23$	**15.** $\frac78$.	**16.** 4
17. $\frac{a^2+c^2}{2c}$	**18** 9	**19.** $\frac{1}{ab}$	**20.** $\frac{a^2+b^2+c^2}{a+b+c}$.
21. $\frac{a}{2}$.	**22** ab.	**23** $\frac{ab+bc-b^2}{a}$.	
24. 5.	**25.** -4.	**26.** $\frac47$.	**27.** 3.

Exercises VIII., p. 74.

1. 2 hours.	**2** 120, 80	**3** 25, 24
4 A, 45; B, 60.	**5** A's share, £4, B's, £6, C's, £480.	
6 42.	**7.** £3.	**8** 60 miles
9 4 miles per hour	**10** £175, £225	**11** £411, rate 10·95%.
12. £7. 10*s*., £9, £7 4*s*	**13.** £5000	**14** 1050.

Exercises IX., p. 81.

1. 13, 11	**2.** 10, 2	**3** 3, 6
4 $x=3,\ y=2,\ z=5$	**5** $\frac{11}{12}, \frac{1}{11}$	**6** 9, 12
7 8, 12	**8** $\dfrac{p-a}{7}, \dfrac{p-b}{4}$	**9** 6, 9.
10. 2, 3	**11** $2\frac{1}{7}, \frac{3}{7}$	**12.** $x=1, y=-1, z=0$

13 0·02, 2·9. **14** am^2, $2am$ **15** $\frac{3}{2}b, -\frac{a}{2}$. **16** a, b.

17. $\dfrac{a^2+b^2}{bef+acd}, \dfrac{a^2+b^2}{bcd-aef}$ **18** $\dfrac{b(b-a)}{a+b}, \dfrac{a(a+b)}{a-b}$.

19 $x=1, y=2, z=3$ **20** $3a, -2b$

21 $x=\dfrac{n(a-b-c)}{a-3b+c},\ y=\dfrac{n(b-c-a)}{a-3b+c},\ z=\dfrac{n(c-a-b)}{a-3b+c}$

22. $x=a,\ y=2a,\ z=3a$

23 (i) $\dfrac{p}{4m^2}, \dfrac{p}{2m}$, (ii) $x=12,\ y=-60,\ z=60$

Exercises X., p. 87.

1. $\frac{31}{43}$	**2.** £11·018 **4.** $\frac{5}{3}$	**5** £10, £5, £1000.
6. 120 lbs.	**7** £1666$\frac{2}{3}$, £1000, £333$\frac{1}{3}$	**8** 76$\frac{1}{2}$, 202$\frac{1}{2}$.
9. 4	**10** £411, 10·95%.	**11** $\frac{3}{10}, \frac{7}{10}$
12 325, 175.	**13.** 9	**14** 7062.

Exercises XI., p. 98.

1. 4, 1	**2** 4, 2.	**3.** $-4, -3$	**4** 2·73, 4·35.
5. 2·74, 3·35.	**6.** 2, -12	**7.** 3, $1\frac{10}{11}$	**8** ± 3

9 $-3\pm\sqrt{44}$. **10** $\dfrac{3\sqrt{34}}{34}$ **11** $\pm\sqrt{\dfrac{(m-n)}{m+n}}$.

12 $0, \pm\sqrt{\dfrac{a^2+b^2}{2}}$. **15.** $\frac{5}{3}, -\frac{3}{5}$. **16** $\frac{3}{2}, \frac{2}{3}, -2, -\frac{1}{2}$.

17. $x=3,\ -1,\ y=4,\ -2$. **18** 4·2426, -14·142.

19 x^2-6x+5. **20** $0,\ 5a,\ -a$. **21** 2·5, -1, $\dfrac{3\pm\sqrt{17}}{4}$.

22. $1,\ 1,\ -1\pm\sqrt{-2}$. **23** $1,\ 2a-1$ **24** $\dfrac{5}{2}\pm\dfrac{\sqrt{65}}{2}$

25. $1+\sqrt{2}\pm\sqrt{(2+2\sqrt{2})}$. **26** $-2+\sqrt{10},\ -2\pm\sqrt{5}$

27 $0,\ -\frac{243}{193}$. **28** $\pm\sqrt{22}-1,\ \pm\sqrt{7}=1$.

29. $\pm\dfrac{\sqrt{4ac+c^2}+c}{2(a+b)}$ **30** 4·3 or -1·376. **31** 20·06 $-$ 1·86.

32 $x=12$ or 3, $y=6$, $z=3$ or 12. **34** 3·408, 1·762

35 $\pm\sqrt{2}$ **36** $+\frac{1}{2}\sqrt{10},\ 0$ **37** $1,\ \dfrac{3\pm\sqrt{5}}{2}$

38. 3·217, 2·233 **39** $\sqrt{3}\pm1=2\cdot732,\ 0\cdot732$.

40 $x^2-14x-351=0$. **42** 3·733, 0·2679

Exercises XII., p. 101.

1 $x=5,\ y=1$ **2** $x=6\frac{1}{3},\ 3,\ y=2\frac{5}{8},\ \frac{1}{2}$.

3 $x=5,\ -\frac{78}{18},\ y=4,\ -\frac{118}{57}$. **4** $x=6\cdot8,\ 4,\ y=-5\cdot4,\ 3$.

5 $x=\pm3,\ y=\pm1$ **6** $x=4,\ 3,\ \mp2\sqrt{6}-6;\ y=3,\ 4,\ \pm2\sqrt{6}-6$.

7. $x=\pm5,\ \frac{3}{4},\ y=3,\ -\frac{5}{4}$. **8** $x=3,\ -4\frac{1}{3};\ y=\frac{1}{2},\ -3\frac{1}{6}$.

9 $x=\pm7,\ \pm\sqrt{51};\ y=2;\ 0$ **10** $x=8,\ y=\pm\frac{5}{2}\sqrt{7}$

11 $x=3,\ 1\cdot5;\ y=-1,\ 5\cdot75$

12 $x=\frac{1}{3},\ -\frac{1}{8},\ y=\frac{1}{4},\ -\frac{1}{7},\ z=\frac{1}{6},\ \frac{1}{28}$.

13 $x=\dfrac{b^2}{\sqrt[3]{b^3-a^3}},\ y=\dfrac{a^3-b^3}{b\sqrt[3]{b^3-a^3}},\ z=-\dfrac{a^3}{b\sqrt[3]{b^3-a^3}}$.

14 $x=\pm5,\ y=4$ or 3, $z=3,\ 4$ **15** $x=1,\ -3,\ y=3,\ -5,\ z=3\frac{1}{2},\ -5\frac{1}{2}$

16 $x=0\cdot5,\ 0\cdot4,\ y=0\cdot4,\ 0\cdot5$

17. $x=\dfrac{\pm\sqrt{33}+1}{\pm\sqrt{60+4\sqrt{33}}};\ \ y=\dfrac{8}{\pm\sqrt{60\pm4\sqrt{33}}}$

Exercises XIII., p. 107.

1 15s. per dozen **2.** 9, 16. **3.** $x=16,\ -3;\ y=3,\ -16$

4 108 yds., 45 yds **5** £100 **6** 5 ft

7 12 and 8. **8** 13 and 7. **9.** 12 in., 27 in.

10 $x=\frac{5}{2}(\pm\sqrt{3}-1),\ y=\frac{5}{2}(\mp\sqrt{3}-3)$ **11.** 27, 54, 81.

Exercises XIV., p. 112.

1. 16, -8, 4 **2** 1, $\dfrac{1\pm\sqrt{17}}{2}$ **3** $-0{\cdot}5$, 1, 0·25

4. $2a$, a. **5** 5, -2, -3 **6** $-3{\cdot}732$, 0·264, 4.

7. 11, -5, -6. **8** 7, $\dfrac{5\pm\sqrt{21}}{2}$ **9** 1·13

10. 2 **11** 2·0945 **12** 2·327

13. 1·44354 **14** 3·942 **15** 4, -2

16 1·73 **17** 1·203, 2·622, $-0{\cdot}825$ **18** 2·012

Exercises XV., p. 150.

1 (i) $R=1{\cdot}42+4{\cdot}66E$, (ii) $E=0{\cdot}295R+0{\cdot}87$;
(iii) $E=0{\cdot}062R+0{\cdot}132$; (iv) $E=0{\cdot}109R+4$

2 $c=8{\cdot}5$, $d=-30$, $F=8{\cdot}5R-30$ **3** 1686

6. $n=1{\cdot}042$, $pv^{1{\cdot}042}=$ const **7** $n=1{\cdot}35$, $c=441$, $p=61{\cdot}05$

8. $a=3{\cdot}25$, $b=0{\cdot}2$, $y=3{\cdot}25+0{\cdot}2x^2$ **9** $c=2{\cdot}6$, $n=2{\cdot}546$

10 102, 14700 **11** $a=2$, $b=0{\cdot}05$, $y=2+0{\cdot}05x^2$

12 $a=2$, $b=-0{\cdot}2$, $c=0{\cdot}05$ **13** 1·2, 3; $y=3x^{1{\cdot}2}$

14. $n=0{\cdot}86$, $pv^{0{\cdot}86}=c$ **16** $l=47{\cdot}6B-300$, 0·52%.

17 $a=0{\cdot}3$, $b=2{\cdot}5$, $y=0{\cdot}3x^2+2{\cdot}5$, 142·4

18 $A=0{\cdot}5$, $b=-1$, $y=0{\cdot}5e^{-x}$

19. (*a*) £3675, (*b*) £2812·5, (*c*) £2860 **20** 7440, 540

21. (i) $a=32{\cdot}26$, $b=-4844$, $n=0{\cdot}94$; (ii) $a=32{\cdot}04$, $b=-7200$

22 $13{\cdot}08h=v^{1{\cdot}8}$ **23** 14790, 14360, 13540

24 $a=2{\cdot}5$, $b=0{\cdot}25$, $n=0{\cdot}35$. **25** $c=7{\cdot}6$, $n=0{\cdot}4229$, $a=0{\cdot}1669$

27 1590 sq. ft **28.** $A^2=a^2+b^2$, $\tan e=\dfrac{b}{a}$

30 values of y are: 2·45, 3·656, 5·453, 8·136, 12·13, 18·1, 26·39, 40·29, 60·12 aver. val. $=17{\cdot}85$, slope at $x=4$ is 4·854

33. 247400 ft.-lbs., 73·4 ft. per sec. **34.** 8 miles per hour

Exercises XVI., p. 161.

1. 3·021, 41° 37′, 53° 23′. **2** 1·798 in., 2·426 in. **3** 255·5

4. 97° 44′, 31° 16′ **5** 519·6, 759

6. 40·44 ft **7** 89·66 yds **8** 1·08318

10. 695 yds., 1·62 min. **11** 117·7 ft.

Exercises XVII., p. 168.

1 $A=60^\circ$, $B=45^\circ$, $C=75^\circ$ 3 $53^\circ\ 4'$, 0·54 sq. ft

4. $55^\circ\ 46'$ 5 $73^\circ\ 24'$. 6 0 5999, $73^\circ\ 44'$.

7 $108^\circ\ 28'$, $38^\circ\ 58'$, $31^\circ\ 34'$ 8. 90°, 210 sq ft 9 0 7670, $74^\circ\ 58'$

10 42°. 11 $41^\circ\ 24'$. 12. $37^\circ\ 22'$.

13 $38^\circ\ 56'$. 14 $50^\circ\ 28'$. 15 $34^\circ\ 8'$, 4114 sq. ft

16 $A=51^\circ\ 54'$, $B=104^\circ\ 44'$, $C=23^\circ\ 22'$ 18 $36^\circ\ 52'$, $53^\circ\ 8'$, 90°.

19. $64^\circ\ 38'$, 2 738 ft. 20. 0 6363 ft 21 5314 sq. ft.

22 1959 sq ft 23 454 1 sq ft 24 28 45 sq in.

25. $67^\circ\ 24'$, $59^\circ\ 28'$, $53^\circ\ 8'$ 26 $29\ 4^\circ$, $31\ 9^\circ$, $118\ 7^\circ$

Exercises XVIII , p. 173.

1 $B=72^\circ\ 37'$, $C=56^\circ\ 3'$ 2 $B=101^\circ\ 29'$, $C=14^\circ\ 11'$ 3 $\dfrac{\sqrt{3}}{12}$

4. $B=79^\circ\ 6'$, $C=40^\circ\ 54'$ 5 $B=74^\circ\ 40^\circ$, $C=45^\circ\ 20'$.

6 $4^\circ\ 55'$, $168^\circ\ 27'$ 7 93°, 27°, 9 54 8 $108^\circ\ 58'$, $6^\circ\ 2'$

9 6 sq ft 10 $72^\circ\ 12'$, $47^\circ\ 48'$ 11 23 68, 826 6, $111^\circ\ 24'$, $36^\circ\ 36'$

12 $97\ 3^\circ$, $28\ 7^\circ$, 595 ft 13 $129\ 5^\circ$, $17\ 5^\circ$

Exercises XIX., p. 175.

1 $68^\circ\ 25'$, 243 3 ft 2 516 3, 3003

3 32 62 ft , 10°, $151^\circ\ 23'$, 138 ft , $28^\circ\ 37'$, $122^\circ\ 46'$.

4 $81^\circ\ 45'$ 5. $43^\circ\ 55'$

6 $c=6\ 68$, $B=125^\circ\ 49'$, $C=1^\circ\ 52'$, $c=196\ 9$, $B=54^\circ\ 11'$, $C=73^\circ\ 30'$

7. $B=51^\circ\ 17'$ or $128^\circ\ 43'$ 8 $B=41^\circ\ 42'$ or $138^\circ\ 18'$

10. $C=62^\circ\ 31'$ or $117^\circ\ 29'$, $A=102^\circ\ 18'$ or $47^\circ\ 20'$

11. $C=45^\circ$ or 135°, $B=105^\circ$ or 15°, $b=\sqrt{3}+1$. 12 $32^\circ\ 26'$.

13. $C=60^\circ$ or 120°, $a=300$ or 86·6, $A=90^\circ$ or 30°

(iii) No $C=90^\circ$, 173 2

Exercises XX., p. 182.

1 105 ft. 2 488 5 ft 3 $BP=241$ ft , $BAQ=29^\circ\ 6'$.

4. 367·8 ft 5. 1701 ft 7. 86 6 ft. 10 988 6 yds

11. 1·152. 12 106 ft 13 229 7 yds 14 114 41 ft.

15 $h=0{\cdot}9292l$ 16 1000 ft 17 37 4 yds 18 73·2 ft.

19. 0 8166 miles 20 1034 ft 21 8769 yds.

22. 56·5 ft , 94 ft. 23 114 ft. 27. 0 4803 l·

Exercises XXI., p. 189.

1	18 yds.	2	£16. 17*s.* 9 6*d.*	3	4 686 ft.
4.	31·11, 62·22	5.	1428 sq. ft.	6	3 ac 1 r.
7.	*374·122* sq ft.	8.	*10* ft *6* in	9	*210 sq in.*
10.	109·81 sq. ft.	12.	3 338, 3 343, mean 3 340 acres.		
13	6 chains, $2\frac{1}{2}$ chains.	14	721721 sq. ft.	15.	892 92 yds.
16	1764.	17	2 576 acres.	18	7 ch. 50 links.
19.	6 ac. 3 r.	20	3·849 yds		

Exercises XXII., p. 199.

1	8 168, 1 3 ft.	2	468 ft	3	112 6 sq. in
4	1·84 ft	5.	2240 14 sq ft	6.	104 7 ft
7.	10·5 ft.	8	143 yds	9	183·26 sq in.
10	20 106 sq. ft.	11	15 187 ft	12	12 in.
13	11 55 ft	14.	333 sq ft	15	£175. 12*s*
16.	22 8 ins.	17	15 ft	18	23·22 sq. in
19.	$a:b=3\ 414\ \ 1$	20	£833 17*s* 3*d*	21	1612 5 sq. ft
22.	2732·4 sq ft	23.	5173 sq. ft	24	1808 sq ft
25	7200	26	19 43 sq in	27	32 78, 32·598.
28	*293·1* sq. ft.	29	*169 85 sq ft*		

Exercises XXIII., p. 205.

1. (i) 402·176 sq. ft , 1608 704 cub. ft ; (ii) 2 125 ft.
(iii) 678·5 lbs. ; (iv) 280 ft

2. 1812 1 cub in , 905 52 sq. in., 21·7

3. (i) 402·2 sq in , 402·2 cub in , 104 6 lbs ; (ii) 4 in ; (iii) 30 ft

4	190·76 sq. ft , 187 1 cub. ft			5.	2 18 lbs
6.	7392 lbs.	7	122·4 lbs	8.	2230 cub in , 598 1 lbs
9.	879 8 sq in., 754 1 cub. in			10	20
11.	3·398 in.	12.	19736640	13.	165·748
14.	15·7 min.	15	95 5 tons	16	53 56 sq. in

Exercises XXIV., p. 211.

1.	10 ft., 400 sq. ft			2	47 124 cub. ft , 52 48 sq. ft.
3.	6 ft.			4	278 6 cub in , 114 8 lbs.
5	*138·5 sq. in., 96 cub. in , 1 83 lbs*				
6.	924 cub. ft.	7	19·7 : 1.	8	£5 10*s.*
9.	14·4 ft.	10.	11315·9 cub. in , 3464·4 sq ft.		
11	173·2 cub. in.	12.	125·7 sq. ft		

Exercises XXV., p. 215.

1 (i) 491 sq in, 1023 cub. in; (ii) 7·444 in.; (iii) 3 385 in.
2 213·6 sq in, 645·4 cub. in **3** 1219 cub. in., 484 sq. in.
4 7·432 ft **5.** 59·57 cub. ft. **6** 648000.
7 1678 cub in **8** 0 5198 in., 0·828 in

Miscellaneous Exercises XXVI., p. 223.

1 38 5 sq in, 9 629 sq in **2** 2130 7 sq ft, 12016 58 cub ft.
3 19 63 lbs **4** 1232 cub ft
5 1256 63 sq ft, 5321 cub. ft. **6** 16 18
7 171 7 sq ft, 249·4 cub ft. **8** 4110 grams
9 151 78 cub ft **10** 1 083 to 1. **11** 10 ft
12 36372 cub ft **13** 4 243 cm. **14** 4 in
15. 2367 lbs. **16.** 16 in. **17.** 11 62 in.
18 1·628 in **19** 11·1 in. **20** 3 5
21 2087 96 lbs **22** 100 6 lbs **23.** 24 25 ft.
24 32 91 sq in **25** 217$\frac{7}{33}$ yds, 7 7 lbs **26** 99 9 sq ft.
27 4 06 in **28** 0 2209 cub in **29** 12 lbs 6 6 oz.
30 3412 lbs **31.** 15 52 ft, 2352 cub ft.
32 50480 cub ft 22 93 ft **33** 10 ft

Exercises XXVII., p. 241.

1 $z = 1\ 732$ **2** 7 071, 64° 54′, 55° 33′, 45° **3** 2·45.
4 7 071, 45°, 53° 8′, 0 4242, 0·5657, 0 7071
5 1 75, 2 082, 1 268 **6** 3 4, 0 5882, 0 4413, 0 6764.
7 8 775, 0·6238, 0 7798, 0 6839.
8 $x = 1\ 348$, $y = 3\ 728$, $z = 3\ 078$
9. 3 776, 0·5041, 0 6101, −0·6101
10 14·45, 39 71, 90 63 **11** 3·283, 0 4568, 0 7004, 0·5483
12 3 624, 9·959, 16 96 **13** 8 55, 23 49, 43 3, 80° 3′, 62°.
14 9·063, 4·226. **15** 5, 53° 13′. **16.** 9·443, 58°.
17 96 59, 25·88. **18** 46·98, 17 1

Exercises XXVIII., p. 261

1. 20·7 lbs., 121° 15′, 4 43 **2.** 39·4, 188° 49′, 114, 277°.
3. 328·5, 101°·3, 2 12; 257, 60°, 1 7
4. 3·9, 61°. **5** 11 35 knots, 12° 15′
6. 30° N of E., 47° N. of W **7** (*a*) 14·5, 73°; (*b*) 23, 27°.

8. $a^2 = b^2 + c^2 - 2bc\cos\alpha$, $a^2 + b^2 + c^2 - 2ab\gamma - 2bc\cos\alpha - 2ac\cos\beta$.

9. 6·75 knots, 21° S. of E.

10. (*a*) S.E., (*b*) 23° E of N., (*c*) N., (*d*) 23° E. of S., (*e*) no wind.

14. 30·47, 173° 52′. **15.** $A = 22\ 4$, $B = 29\ 6$. **16.** $\alpha = 49°$, $\beta = 141°$.

17. $C = 4,\ 46$, $\gamma = 2\ 5°$, 80° **18** 25, 45° ; 24·2, 2° 36′.

20. 4·368, 76° 42′. **21.** 1 8, 55° 18′.

22. 27, 141°. **24** 14·6, 161° 30′, 4·534.

25. 6 ft. per. sec. − 210 f s s **26** 24 2, 2° 36′.

27. (*a*) 6000 ft.-lbs per sec, (*b*) 2645 ft.-lbs per. sec., (*c*) 0, (*d*) − 1060 ft -lbs. per sec.

28. $A = 22{\cdot}5$, $B = 30\ 4$.

29. 2·035, 7 5° W of S.; 5 77, 25° E of N ; 6 5, 11·5° W. of S. $A\,.\,B = 2{\cdot}472$, $AC = 2\ 863$

30. $\theta = 60\ 3°$, $60\ 7°$, $81°$; 66 54, $\alpha = 107°\,38'$, $\beta = 69°\,14'$, $\theta = 27°\,28'$.

Exercises XXIX., p. 269.

1. $-62\frac{1}{2}$. **2.** 0. **3** 13 5 **4** $22\frac{2}{3}$

5. 9, 8, 7 . **6** 6, 8, 10, or 10, 8, 6 **7.** 10

8. 25 **9** $\frac{3}{4}$ **10** $18\frac{16}{27}$, $\frac{29}{27}$. **11** 973.

12 $\frac{2(2a+d)}{3d}$, $\frac{2a+d}{3d}$. **13** 62 **14.** 1, 3, 5 ....

16. 20 **17.** 5. **18.** $a = 10$, $d = -2$. **19** 77.

Exercises XXX., p. 273.

1. $86\frac{1}{4}$ **2** $18\left\{\left(1 - \frac{1}{3}\right)^{10}\right\}$ **3** $-0{\cdot}592\left\{\left(\frac{3}{10}\right)^{10} - 1\right\}$.

4. − 185. **5** 9780. **6** 45920 **7** 80.

8. $16\left\{1 - \left(\frac{1}{4}\right)^{10}\right\}$ **9** − 16 7728 **10** − 136·5.

11. 18, 54, 162 , or − 18, − 54, − 162, etc. **12** 4, 8, 16, 32, 64.

13. 1, 4, 16. **14** $-\frac{211}{8}(\sqrt{3} - \sqrt{2})$. **16** impossible ; > 1.

17. $\frac{3}{5}$. **18** 9. **19** $74\frac{2}{3}$.

20. $\frac{3}{2}$. **21** 16, 24, 36 . . **23** $r = \pm 2$, $a = 3$.

Exercises XXXI., p. 275.

1. $2\frac{2}{5}$, 3, 4, 6 **2.** 5, 4, 3·2 **3** 7. **4.** 5.

5. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ **6** 4, 16. **7.** 1, $\frac{6}{5}$, $\frac{3}{2}$. **8** 18.

9. $2\frac{2}{11}$, $2\frac{2}{5}$, $2\frac{2}{3}$. **10.** $\frac{13}{4}$, 3, $\frac{36}{13}$. 2, $\frac{13}{4}$, $\frac{9}{2}$; 2, 3, $\frac{9}{2}$; 2, $\frac{36}{15}$, $\frac{9}{2}$.

Exercises XXXII., p. 277.

1. $38{\cdot}4$. 2. $57{\cdot}6$ 3 $\frac{3}{2}$. 4 $\frac{16}{7}$.

5 $100{\cdot}8$ 6 $74\frac{2}{3}$ 7. $\frac{4}{9}$ 8. $\frac{14}{3}$.

9. $\frac{14}{3}$. 11 $-\frac{19}{6}$, $\frac{16}{3}\times\left(\frac{15}{3}\right)^3$, $1\frac{7}{33}$.

12 -154, -148, -142 13 $22\frac{2}{3}$, $22\frac{2}{3}$.

14. -70, 110, 290, 470, 650, 830, 1010, 1190

15. $r=1{\cdot}5$, 768, 1152, 1728, etc. 16 $4n(n+1)$, $(2n+1)^2$.

17 (a) $\dfrac{x^{n}-1}{x-1}$; (b) $\dfrac{2^{n+1}x^{n+1}-1}{2x-1}$; (c) $\dfrac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$.

19. $y^2\times\dfrac{y^{2n}-1}{y^2-1}+bn(n+1)$

Exercises XXXIII., p. 287.

1. -4 2. $\frac{1}{3}$ 3 5 5. $\dfrac{2048}{675}x^3$.

6 $55a^9b^2$, $462a^5b^6$, $462a^6b^5$.

7 $x^6\pm6x^5a+15x^4a^2\pm20x^3a^3+15x^2a^4\pm6xa^5+a^6$

8 $625-200x+2400x^2-1280x^3+256x^4$.

9. $1820x^{12}a^4$ 10 $2-\dfrac{1}{3\,.\,2^2}-\dfrac{1}{3\,.\,6}-\dfrac{}{2^4}$; 1 913.

11. $a^2+6ax+x^2+(4a+4x)\sqrt{ax}$

Exercises XXXV., p. 308.

1. $4x^3+9x^2-2x$ 2 nAx^{n-1} 3 $a\cos ax$.

4. $Aa\cos ax$ 5 $-Aa\sin ax$ 6 $\frac{3}{2}\sqrt{x}$

7. v_0+at 9 $\cos x$; $-\sin x$, $\sec^2 x$

11. $-ab\sin(bx+c)$; $\dfrac{b}{a+bx}$ 12 $-\dfrac{x}{\sqrt{a^2-x^2}}$.

13. $-\operatorname{cosec}^2 x$ 14. $\dfrac{1}{x}$. 15 $a^x\log_e a$.

16 $nax^{n-1}\cos ax^n$ 17. $-\dfrac{a}{t^2}$. 18. $-\dfrac{t}{\sqrt{a^2-t^2}}$.

19 $\dfrac{2}{x}$ 20. $8x+13$ 21. $10x-9$. 22. $5x^4+12x^2$.

23. $-3x^{-\frac{5}{2}}$. 24. $-1{\cdot}408cv^{-2{\cdot}408}$. 25 ft. 26. f.

Exercises XXXVI., p. 322.

1. $14x$ 2. $3\cos x$ 3. $-3\sin 3x$ 4. $-10\sin(2x+3)$.

5. $\frac{1}{x}$. 6. $\frac{3A}{x}$ 7. $6e^{2x}$ 8. $-kAe^{-kx}$.

9. $6t-4$. 10. $2At+B$ 11. $12\cos(4t+9)$.

12. $-63\sin 2(6t^3+9t+5)\times(2t^2+1)$ 13. $\frac{14}{8}e^{\frac{t}{8}}+72\cos 8t$.

14. $11e^t\sin(6t+7)+66e^t\cos(6t+7)$

15. $Abe^{bt}\sin(ct+f)+Ace^{bt}\cos(ct+f)$

Exercises XXXVII., p. 334.

1. $40x^3$. 2. $\frac{8a^2x^3-4x^5}{(a^2-x^2)^2}$ 3. $\frac{2x(a-2x^3)}{(a+x^3)^3}$.

4. $\sec^2 x$. 5. $-\frac{1+x-x^2}{(1+x^2)^{\frac{3}{2}}}$ 6. $\frac{mq-pn}{(px+q)^2}$, $-\frac{n}{x^{n+1}}$.

7. $\frac{nx^{n-1}}{(1+x)^{n+1}}$, $x^{a-1}(a\log x+1)$. 8. $\frac{2(1-x^2)}{(1+x^2)^2}$

9. $e^{\sin x}\cos x$ 10. $\frac{\log_a e}{\sin^{-1}x\sqrt{1-x^2}}$ 11. $-\frac{x\sin\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}$.

12. $\frac{x\cos\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}$ 13. $\frac{x}{x^2+a^2}$ 14. $\frac{2x}{\sqrt{1-x^4}}$

15. $3x^2+2x+1$ 16. $\frac{2}{1+x^2}$ 17. $\frac{1}{1-x^4}$.

18. $\frac{2a}{x^2-a^2}$ 19. $-\operatorname{cosec}^2 x$ 20. $\frac{1}{\sqrt{a^2+x^2}}$

21. $\frac{2}{1+x^2}$ 22. $-3x\sqrt{a^2-x^2}$ 23. $\frac{2x(2-x^2)}{\sqrt{x^2-1}(x^4-x^2+1)}$.

24. $\frac{m\cos(m-1)x}{(\cos x)^{m+1}}$. 25. $\frac{1}{1+x^2}$ 26. $(4bx+3a)x^2$.

27. $\frac{x}{\sqrt{x^2+a^2}}$ 28. $2\sin x\cos x$ 29. $\sin^2 x(3\cos^2 x-\sin^2 x)$.

30. $2(a+2x)(ax+x^2)$. 31. $e^x(\cos x-\sin x)$ 32. $x^{c-1}(\log_a x^c+\log_e a)$

33. $\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$. 34. $-\frac{1}{x\sqrt{x^2-1}}$. 35. $\frac{1}{1-x^2}+\frac{x\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$.

Exercises XXXVIII., p. 351.

1. 10·4, 10·004, 10·0004, 10 . 2. $5+4{\cdot}2t$; 26.

3. $150-10t$; 80 f.s.; -10 f.s.s., 31·06 lbs. 4. 1·6 miles.

5 (i) 52·01, (ii) 50·201, (iii) 50·0101; 50 ft. per sec.

7. 3·4 f.s.s., 1·419 lbs. 8. 14·26 f.s., 4 f.s.s.; 12·43 lbs.

14 (i) 0·96, 2·9, 43·6, 54·0 f.s.; (ii) 62·48, 57·61, 41·61 f.s.s.

Exercises XXXIX., p. 370.

1. Each 16 2 $x=0$ 3 $x=-1$ max., $x=+1$ min.

4 6·25 sq. ft. 5 Line is bisected 6 Max. none, min. $=-64$.

7. Max. 6, min. $1\frac{10}{27}$ 8 Max. 0, min. $\pm a$

10. (i) 8, 4; (ii) 9, 3 11 $y=\pm 1$ 12 $x=\sqrt{\frac{a}{b}}, 2$

13 (i) $x=0$ max.; (ii) $x=3$ min.; (iii) $x=0$ min.

14 $x=0$ max. $=2\sqrt{a}$ 15 $h=r=147·1$ ft.; area $=203907$ sq. ft.

16 2·55 cub. ft. 17 $\frac{1}{2}\left(\frac{\pi}{2}+a\right)$, $\frac{1}{2}\left(\frac{3\pi}{2}+a\right)$ 18 $3\frac{1}{3}$, 4.

19 $\frac{a(5+\sqrt{13})}{6}$ max., $\frac{a(5-\sqrt{13})}{6}$ min. 20 722 sq. ft.

22 Each side $=\frac{c}{\sqrt{2}}$ where c is the length of the hypotenuse.

23 (i) 1 max., 3 min.; (ii) $\frac{8a^2}{3\sqrt{3}}$ max., 0 min.

Exercises XL., p. 376.

1 $12x^2+18x-2$; $24x+18$ 2 $a\cos ax$, $-a^2\sin ax$

3 $Aa\cos ax$, $-Aa^2\sin ax$ 4 $-Aa\sin ax$, $-Aa^2\cos ax$, or $-a^2y$.

5 $\frac{3}{4}\frac{1}{\sqrt{x}}$. 6 a.

Exercises XLI., p. 387.

1 $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots$.

2 (i) $\log a+\frac{x}{a}-\frac{1}{2}\frac{x^3}{3a^3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5a^5}+\ldots$, $2^n\left\{1+\frac{nx^2}{2!}+n(3n-2)\frac{x^4}{4!}\right\}$

3 $x^4+\frac{4}{3}x^6+\frac{6}{5}x^8+$ etc. 4 $x^x(1+\log x)$; $x^x\{(1+\log x)^2+x^{-1}\}$

5. $e^{\tan x}(1+x\sec^2 x)$; $e^{\tan x}\sec^2 x\{2+x(\sec^2 x+2\tan x)\}$

6. $\frac{1}{1+x^2}$; $\frac{-2x}{(1+x^2)^2}$. 7 $2(-1)^{n-1}\frac{(n-3)!}{x^{n-2}}$

8. $e^x\{x^3+3nx^2+3n(n-1)x+n(n-1)(n-2)\}$

9. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+$

10. $\sin^{-1}x+\frac{h}{(1-x^2)^{\frac{1}{2}}}+\frac{x}{(1-x^2)^{\frac{3}{2}}}\frac{h^2}{2!}+\frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}}\frac{h^3}{3!}+ \quad .$

11. $x+\frac{x^2}{2!}+\frac{2x^3}{3!}+\frac{9x^5}{5!}+\dots$. **12.** $x-\frac{x^3}{3!}+\frac{x^5}{5!}$

Exercises XLII., p. 417.

1 21, 1·0987.

2. $\frac{x^3}{3}$; $\frac{1}{b}\sin bx$, $\log v=\log 3=1·0987$; $\frac{x^3}{3}=\frac{1728-729}{3}=333.$

3 $\Big[2cx+x^2\Big]_{10}^{20}=560-180=380$

4 $\Big[(c+nx^2)^3\Big]_a^b=436^3-132^3=80581888$ **5** $\frac{1}{a}\sin ax$

6. $\frac{1}{a}\tan ax$ **7** $\frac{1}{a}\tan^{-1}ax$ **8** $y=\frac{l^{ax}}{a\log_e l}$.

9 $\frac{A}{b}\sin(a+bx)$ **10** $\frac{1}{b}\tan^{-1}(a+bx)$

11 $\frac{1}{3q}(p+qx)^3$ **12** $\frac{1}{b}\sin^{-1}(a+bx)$

13. $\frac{a}{m+1}x^{m+1}$; $ax+\frac{b}{n+1}x^{n+1}$, $\frac{\sin(a+bx)}{b}$, $\log x$, $\frac{1}{b}\log(a+bx)$

14 $\frac{1}{a}\tan^{-1}\frac{x}{a}$ **15** $-\frac{1}{2}\log(a^2-x^2)$

16. $\frac{1}{a}\sec^{-1}\frac{x}{a}$ **17** $\frac{q}{p+q}x^{\frac{p+q}{q}}$

18. $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right)$ $\left(\text{Hint, put } \frac{x^2}{a^2}=z\right)$ **19** $\log\sqrt{\frac{(x+3)^3}{x+1}}$

20. $\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\theta\right)$ (Hint, put $\tan\theta=\phi$ and then split into two fractions.)

21. $\log(\theta+\sin\theta)$. **22** $\frac{1}{5(a^2-x^2)^{\frac{5}{2}}}$.

23. $\frac{2}{3}\tan^{-1}\frac{\tan x}{2}-\frac{x}{3}$. (Hint, divide into two factions.)

24. $\frac{x^6}{6}$; $\frac{2}{3}x^{\frac{3}{2}}$; $\frac{4}{3}x^{\frac{3}{2}}$; $\frac{3}{5}x^{\frac{5}{3}}$

25. $\frac{2}{3}x^3+\frac{3}{2}x^2+5x.$

26. $-\cos x+\sin x.$

27. $-\frac{1}{6}\cos 6x-\frac{1}{2}\cos 2x.$

28. $-\frac{1}{12}\cos 6x+\frac{1}{4}\cos 2x$

29. $\frac{1}{2}\sin 2x-\frac{1}{6}\sin 6x$

30. $\frac{1}{6}\sin 6x+\frac{1}{2}\sin 2x$

31. $\frac{1}{a}e^{ax}$

32. $-\frac{a}{0.37}v^{-0.37}$

33. $\frac{1}{3}at^3+\frac{1}{2}bt^2+ct+g.$

34. $\log\{x+\sqrt{x^2+a^2}\}$ (Hint, put $z=x+\sqrt{x^2+a^2}$.)

35. $\frac{a^{m+x}}{\log a}.$

36. $\frac{1}{a}\log\frac{x}{\sqrt{x^2+a^2}+a}.$

37. $\frac{1}{3}(1+x^2)^{\frac{1}{2}}(x^2-2)$ (Hint, put $z^2=1+x^2$.)

38. $\frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}.$

39. $\frac{2bx-a}{2a^2x^2}-\frac{b^2}{a^3}\log\frac{a+bx}{x}.$ $\left(\text{Hint, put } z=\frac{1}{x}\right)$

40. $-\frac{1}{3}(1-x^2)^{\frac{1}{2}}(x^2+2)$

41. $\frac{1}{2}(x-\sin x\cos x).$

42. $x+\frac{3}{4}\log\frac{x-2}{x+2}$

43. $\frac{1}{2\log_e 2.4}2.4^{2x}.$

44. $\log\tan\frac{1}{2}\left(\frac{\pi}{2}+x\right)$

45. $\log\tan\frac{x}{2}$

46. $y=1.25x^2$; vol. $=32180$

47. 18682 cub. in.; 10.8 cub. ft.

Exercises XLIII., p. 439.

1. 2.2″, 2.983
2. 21″
3. 1778
4. $4\frac{1}{2}$″, 241.5
5. 8.9 sq. in.; 1.9″, $I=22.5$, $k=1.59$

Exercises XLIV., p. 464.

1. $-\frac{1}{2}\left\{\frac{\sin(a+b)x}{a+b}-\frac{\sin(a-b)x}{a-b}\right\}$

2. $x^2\sin x+2(x\cos x-\sin x).$

3. $\frac{1}{(a-b)}\{a\log(x-a)-b\log(x-b)\}$

4. $x+\log\frac{x-3}{x-2}.$

5. $2\tan^{-1}x-\frac{1}{2}\tan^{-1}\frac{x}{2}.$

6. $\log\{\sqrt{x+2}\times\sqrt[4]{x^2+4}\}+\frac{1}{2}\tan^{-1}\frac{x}{2}$

7. $\frac{x^4}{4}\left\{(\log x)^2-\frac{1}{2}\log x+\frac{1}{8}\right\}.$

8 $x+\dfrac{a^3\log(x-a)}{(a-b)(a-c)}+\dfrac{b^3\log(x-b)}{(b-c)(b-a)}+\dfrac{c^3\log(x-c)}{(a-c)(b-c)}$.

9 $\dfrac{2}{25}\log\dfrac{x-3}{x+2}-\dfrac{3}{5(x-3)}$. **10** $\sin\theta-\theta\cos\theta$

11 $x^3\sin x+3x^2\cos x-6x\sin x-6\cos x$. **12** $\dfrac{7}{2}\dfrac{1}{x+1}+\dfrac{11}{4}\log\dfrac{x+1}{x+3}$

13. $5\log(x+1)-2\log(x+3)+3\log(x-4)$.

Miscellaneous Exercises XLV., p. 496.

1 $y=e^{-\frac{7}{2}x}\left(Ae^{\sqrt{\frac{141}{4}}x}+Be^{-\sqrt{\frac{141}{4}}x}\right)$.

2 $y=Ae^{\frac{2}{3}x}+Be^{-9x}$ **3** $y=e^{3x}(Ax+B)$.

4 $y=e^{-\frac{17}{4}x}\left(A\sin\sqrt{\dfrac{31}{16}}x+B\cos\sqrt{\dfrac{31}{16}}x\right)$

5 28·41 **6** 0 662, 3584 lbs. **7** 9 922 **8** 0 733

9 $y=\dfrac{x^3}{3}-\dfrac{x^2}{2}+x+\frac{7}{6}$, $8\frac{2}{3}$ **10** $x=+0\ 6324,\ 2\ 929,\ 2\ 315$

11 $x=8a$; $y=\pm4a\sqrt{2},\ -2a$ **12** 20, 10

13 $y=3x+\dfrac{x^2}{2}$; 1368 **14** $\frac{5}{24}\pi r^3$ **15** 2 1295, 10 22

Examination Paper. 1901.

1 7·446, 0 01254, 5 68, 1546

2 1 69, $-$0 2987, $-$0 000000001387

3 $E=22\ 5$, $R=28$, $P=£5\ 25$, 230

4 0 86 **5** 6 cub ft

6 $A^2=a^2+b^2$, $\tan e=\dfrac{b}{a}$ **8** 329 8, 0 12 % in excess

9. 1·348, 3 702, 3 078, $\alpha=74°\ 22'$, $\beta=42°\ 14'$

11 9 25 f s s , 57·5 lbs

12 nax^{n-1}, abe^{bx}, $ab\cos(bx+c)$, $-ab\sin(bx+c)$, $\dfrac{1}{x+b}$

13. $\dfrac{cv^{0\cdot2}}{0\cdot2}$, $c\log_e v$ **14** $\dfrac{\pi m^2(b^2-a^2)}{2b}$

15 $h=\dfrac{1}{\gamma-1}(-sp+\gamma p)$ or $h=\dfrac{p(\gamma-s)}{\gamma-1}$

16 $T^2=1318h^3$, $V=1271h^{\frac{3}{2}}$. **17** $\delta T=0{\cdot}0037h^{\frac{1}{2}}$ tons per in.

18. $n=260$, $R=0{\cdot}881$

Examination Paper. 1902.

1 3 123, 1704, 1 722, 0 0198

4. (*a*) £3675, (*b*) £2812 5, (*c*) £2860.

5 $c=5\ 56+\dfrac{0\ 06}{f}$; $U=0\ 18C-0\ 15$

6. $P=31{\cdot}6v^{1\cdot78}$ 7 $r=3\ 5$

8. (*a*) 8 148″, 4 644″, 3·64 %, (*b*) 44·43 cub in.

9

k	0 8	0 9	1	1 1
v	7 476	5 98	5 0	4 32

11. 3 905, $\phi=36^\circ\ 52'$, $\theta=39^\circ\ 48'$, $\alpha=51^\circ\ 12'$, $\beta=67^\circ\ 24'$

12. $x=0\ 5282$

13. $V=3\sin 600t+2\ 4\cos 600t$, $C=10$ and -10, $V=3\ 84$ and $-3{\cdot}84$.

14. $y=4\ 353$, $y=1\ 22x+0\ 49$

Examination Paper. 1903.

1 284 7, 2817, 4 5710, 1 5710, 2 5710, 3 339, 1·93, 1768000, 11 03

2 (*a*) 3596, (*b*) $-9\ 897$, (*c*) 32·02

3 $a=1{\cdot}24$, $b=0\ 598$, $n=2\ 353$, 3 593

4 33420, 304 5 1 22 6 $14\ 78\times 10^6$.

7 Aver rates A, 2 8, B, 2 4, A's age $11\frac{1}{2}$ years, 4 2, 1·5.

9 $n=1\ 11$, $c=553$, $x=22\ 39$

10 (*a*) $0\ 4\pi^2$, (*b*) 1687 sq in

11 $OP=3\ 09$, $OQ=2\ 48$, 1 15, 24 2°

12 $\dfrac{M(R_0^2+R_1^2)}{2}$, $M\left(\dfrac{R_0^2+R_1^2}{2}+R_1^2\right)$.

13. $C=10$ and -10, $V=3\ 84$ and $-3\ 84$.

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